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GODFREY'S INVENTION OF THE REFLECTING QUADRANT

BY

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The history of scientific discovery and invention contains numerous instances where an original idea has occurred simultaneously to two men but, for one reason or another, all credit for the discovery has accumulated about one man, while the other has been allowed to sink into an undeserved obscurity. John Hadley, an English country gentleman, is exclusively awarded the honor of having invented the reflecting quadrant, and the rival claimant, Thomas Godfrey of Philadelphia, is virtually unknown.

In seeking a reason for Godfrey's neglect we may find it arises from one of two causes, or a combination of them. First, Britain was rising to a position of maritime supremacy among nations, and London had a number of accomplished instrument makers, factors which made it natural for mariners to prefer London-made navigation instruments; whereas America had not yet produced instrument makers of note. Second, the personal characteristics of the two inventors may have influenced posterity in its judgment. Hadley had an unblemished reputation, the support of influential friends, and encouragement from the members of the Royal Society, of which he later became a Vice-President. Godfrey, on the other hand, was an unknown artisan, reputed to have been an argumentative fellow with a fondness of strong drink, and with few influential friends to advance his cause.

This is neither the time nor the place to attempt a rehabilitation of his character, but it is permissible to undertake the establishment of Godfrey's claim to have been an independent inventor of the reflecting quadrant.

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The years 1730–1732 were notable for the activity displayed in attempts to improve the navigator's instruments. The mariner's outfit at the time comprised: meridian, azimuth and amplitude compasses, sea quadrants, sinical quadrants, cross-staffs and backstaffs. The last mentioned had been invented in 1594 by Captain John Davis, and remained the favorite instrument of British and American navigators well into the 18th century. The backstaff was fitted with a reflector and was the first rough idea of the principle of the modern quadrant and sextant, but the notable feature of all the instruments mentioned was, that to measure an angle at sea the navigator had to depend either on a plumb-line (the quadrant) or he had to look two ways at the same time (the cross-staff and backstaff).

This, then, was the situation when Grandjean de Fouchy in France, John Hadley in England, and Thomas Godfrey in America engaged upon the task of producing an instrument which would bring the reflection of one object to coincide with the direct image of another. De Fouchy and Hadley may possibly have been encouraged upon their task by a paper written by Pierre Bouguer which, in 1729, had obtained for him a prize of the Academy of Sciences in Paris, for the best method of taking the altitude of stars at sea. This was no more than a modification of the backstaff, but the prominence given to it may have awakened the interest of ingenious men who could see in it the nucleus for a broader conception.

Bouguer's paper was not responsible in any way for encouraging Thomas Godfrey. It is improbable that he knew of the existence of the Academy of Sciences. How he became interested in the problems of the navigator is something of a mystery for there is no record of his ever having been on the sea at any time. In fact there is very little known about him at all, but as he was certainly the inventor of a reflecting quadrant, what little is known of him and his invention deserves to be communicated to a wider audience.

Thomas Godfrey was born in Bristol Township, Pa., in 1704. His father, a maltster, died before the child was a year old. Whatever formal education he may have secured can only have been of the scantiest order, for the young Thomas was apprenticed to a glazier in Philadelphia when he reached the age of twelve years. From all accounts his employer was an indifferent worker but Thomas learned enough about his trade to enable him by a natural talent to engage in some superior work, such as glazing and plumbing at the State House. Possibly, too, his employer's lack of enterprise permitted the apprentice to have leisure which was turned to good account in repairing some of the deficiencies in his education.

It was about this time the boy chanced upon his first book relating to mathematics and was fascinated by it. Having absorbed all the book had to offer, he proceeded to borrow every other work on the sub-

ject that came within reach. The available literature cannot have been extensive and what would have discouraged other students of his age was the frequency with which he encountered Latin terms in his reading. Undismayed by these obstacles and determined to pursue the study of mathematics, he contrived to learn enough Latin grammar and syntax to be able to read the few books in that language which came his way.

Inevitably a man of such consuming desire to extend his knowledge would seek out similar characters, and Godfrey became a member of the Leather Apron Club, or Junto, that club of ambitious young men whom Benjamin Franklin gathered around him in 1727. The acquaintance of Franklin and Godfrey continued for, in the following year, when the former set up a printing shop with Hugh Meredith as his partner, he rented the upper floor of the house on lower Market Street to Godfrey and his wife, with whom the printers were to board. The glazier shared the shop on the ground floor where, we may presume, some animated discussions took place, for Godfrey was a great talker.

Godfrey cannot have been a run-of-the-mill glazier. At a quite early age he must have acquired a reputation as a superior workman, otherwise he would not have received commissions to install glass in such places as the State House, Andrew Hamilton's home at Bush Hill, and Logan's beautiful house, Stenton.

James Logan was to prove a staunch friend and but for him it is probable Godfrey's quadrant would have passed into oblivion. He was the most influential man in Pennsylvania at the time. He had been successively secretary to William Penn, Chief Justice, and President of the Provincial Council. A member of the Society of Friends and a wealthy merchant, Logan was a man of great intelligence, and correspondent of numerous men of science at home and abroad. He had the finest private library in America.

While Godfrey was engaged in glazing Logan's windows he must have given the library a surreptitious exploration for, as we shall see, he had located there a copy of Newton's *Principia* which he wanted badly to borrow. This was a treasure indeed, for this copy and one in possession of Governor Winthrop were probably the only copies of the book in America. Although Logan had obligingly lent books to John Bartram, the famous botanist, he was naturally hesitant to lend his copy of Newton to a glazier, but when he found that Godfrey had, by self-instruction, acquired a knowledge of mathematics and astronomy and had sufficient acquaintance with Latin to read a book in that language, he relented and allowed Godfrey to borrow the book.

Mastering the work of the supreme scientist was no mean accomplishment, for there were neither translations nor commentaries to help him over the tough passages. But Godfrey did master it, thereby earning Logan's respect. An intimacy developed between this strangely assorted pair. When Franklin proposed founding the Library Com-

pany, the first public subscription library in the country, Godfrey was selected to solicit Logan's assistance in the selection of suitable books. It is also believed that Godfrey was responsible for persuading Logan to change his attitude toward the Junto, which he had suspected of disloyalty to the Penn family.

The only recorded comment upon Godfrey's character appears in Franklin's *Autobiography*. It should be observed that Franklin was the least censorious of writers and only spoke in mild terms when much more severe criticism would have been justified. He describes Godfrey as "a self-taught mathematician—great in his way" but, he adds, "he knew little out of his way and was not a pleasing companion as, like most great mathematicians I have met with, he expected precision in everything and was ever denying and distinguishing upon trifles, to the destruction of all conversation." [2]

The reader will understand how the garrulous Franklin, whose nature was inclined to expand in the society of his friends, must have been irritated by the interruptions to the conversation by this man who demanded niggling precision in place of generalizations, no matter how brilliant they may be. There were ups and downs in the acquaintance between the two men. It was threatened at one time with irrevocable rupture when Franklin incurred Mrs. Godfrey's wrath. He had engaged the lady's good services as a marriage broker, but as the young printer required his prospective parents-in-law to mortgage their home in order to provide their daughter with a dowry that would pay off his business debts, negotiations fell through and the offended Godfreys left the Market Street house in dudgeon.

The two men patched up their differences, however, and when Godfrey determined to put his knowledge of mathematics and astronomy to practical use in compiling an almanac, he took it to Franklin to be printed for the issues of 1729, 1730, and 1731. This almanac was not an impressive affair as it comprised only a single sheet "in the London Manner." He was apparently informed of Franklin's intention to publish *Poor Richard's Almanac* in 1733 and ceased to prepare his own, for he must have realized he could not compete with the more sprightly "Richard Saunders" whom Franklin invented as his philomath. However, Godfrey did prepare the astronomical material for Bradford's *Pennsylvania Almanac* for the years 1733–1736.

When the American Philosophical Society was organized in 1744, Godfrey was the representative for mathematics. At this time he was known to have invented the reflecting quadrant. It would be interesting to know how he became acquainted with the difficulties experienced by mariners in getting accurate readings with the instrument they used for "shooting" the sun or Polaris. Not all his visits to taverns were

[2] "Benjamin Franklin's Writings," edited by A. H. Smyth. New York, The Macmillan Co., 1905–1907, Vol. 1, pp. 299–300.

associated with glazing or plumbing. Presumably the deficiencies of the Davis quadrant or backstaff were communicated to him by convivial mariners who, in discoursing upon the incidents on their voyages, disclosed the inadequacies of the instrument. These could be pronounced, since it is recorded some navigators in the West Indies had missed their aim at such a conspicuous target as the island of Barbadoes. One can readily understand how this would be a challenge to a man who sought precision in all things.

The special merit which recommended the Davis quadrant dwelt in the relief it afforded from having to look directly at the sun when measuring its altitude, which had to be done with the ordinary quadrant and plumb-line and the cross-staff. The instrument comprised, in addition to its forty-inch staff, a quadrant laid across the staff and divided into two arcs (or "arches"), one of 60° and the other of 30°. Standing with his back to the sun, the observer sighted the horizon through a vane on the shorter arc and a slit in a vane at the end of the staff. When the horizon was correctly sighted, the observer adjusted a shade vane on the 60° arc until this brought the sun's shadow into coincidence with the slit through which the horizon was sighted. The sun's angle could then be read.

During the 125 years succeeding its invention the backstaff had come into general use under a variety of titles, such as the mariner's bow (because its shape bore a mild resemblance to the soldier's crossbow), the English quadrant, the double quadrant, and Davis' quadrant. A number of useful refinements had been added as they became available. When the value of the Vernier's scale was appreciated it was introduced on the instrument to secure a more precise reading. William Gascoigne contributed the telescopic sight, and the tangent screw was applied to provide a slow and steady adjustment as the desired point was approached. Thus, it will be seen the mathematical practitioners had given serious thought to improving the instrument.

But with all these refinements, the backstaff had its faults. It was notably defective when the sun rode high overhead, as it does in the low latitudes, and it was this particular defect which appears to have engaged Godfrey's attention. This seems all the more probable as it must have been the subject of conversation among those mariners engaged in the West Indies trade whom he was likely to meet in the taverns he frequented.

Godfrey's first effort by which he proposed to determine a star's or the sun's altitude was incorporated in an instrument by which he sought to bring the star to the horizon in one reflection. This was unsatisfactory as the slightest motion caused the two objects to fly apart. As the deck of a ship is not the most stable base, he arrived at the conclusion this was not the solution to the problem.

However, notice must be taken of this instrument as it establishes

Godfrey's preoccupation with the subject prior to any of the dates affixed to the documents in the subsequent correspondence relating to the second instruent.

Dissatisfied with this accomplishment and wishful to better it, he is said to have had a stroke of inspiration while working at Stenton. The story probably belongs to the Apocrypha of Science along with that of Newton and the apple, but we present it for what it is worth. While working in the house, Godfrey was momentarily blinded by a flash of light which was found on investigation to come from a piece of glass he had let fall outside and which had caught the sun's rays. Pondering upon the phenomenon of the reflection of light, he is alleged to have been seized with an inspiration on the solution of his quadrant. To verify the theory he had formed he went to the library to see what Newton had to say on the subject and, the story adds, it was at this moment Logan entered the room. The question as to what he was doing there elicited the explanation of Godfrey's interest in astronomy, and this led to an appreciation of their respective qualities, and the loan of the book to Godfrey.

With an increased knowledge of the laws of reflection Godfrey now designed an instrument in which he employed two reflections of the stars. This, he found, answered his purpose admirably in theory. He also learned that the angular distance between the two objects was double the angle of inclination between the two reflecting planes, thereby learning the underlying principle of the modern sextant. This was all revealed to James Logan in June, 1730.

Now we arrive at the period when the dates of his accomplishment are established by affidavits sworn to before Samuel Hasel, mayor of Philadelphia, and recorded under date of August 27, 1733, and by James Logan's correspondence with the Royal Society.[3]

In an affidavit sworn by George Steward, mate, and John Cox, master of the sloop *Trueman*, the former affirmed that Godfrey had revealed to him at about the date assigned in Logan's correspondence, June 30, 1730, he was contemplating the construction of an instrument that would show the altitude of the sun at sea "or the distance of any one star from another, or from the moon," by two reflections. Godfrey then proceeded to show on paper how he proposed to execute the idea. Steward was so favorably impressed by the idea that he entrusted his backstaff to Godfrey to have the necessary parts added.

A second affidavit sworn to on the same day affirms that Edmund Wooley, a carpenter of Philadelphia, was entrusted with the task of making the additions to this instrument of Steward's. Following Godfrey's drawing, Wooley made a contrivance that slipped over the main staff in place of the horizon vane, making a reflecting instrument with a limb of 30° of arc. Using the original scale on the arc it was necessary

[3] *Philosophical Transactions* of the Royal Society, Vol. VII, pp. 441–450 (1733–1734).

to double the readings to determine the correct angle of observation. Provision was made for rotating the index mirror through 20° or more from the zero position when angles greater than 60° had to be taken. Readings in this case were increased by the angle of rotation of the index mirror, and the sum was doubled to obtain the observed angle.

The construction of the necessary parts was quickly performed and the completed instrument was shown to James Logan in November, 1730. It was Logan's idea that the instrument should be put to a practical test by someone with wider experience of measuring angles than an ordinary mariner might be assumed to possess, and at his suggestion Joshua Fisher made use of it on his survey of Delaware Bay then in progress. It is affirmed he fixed the location of Cape Henlopen on his map of Delaware Bay[4] with the aid of this instrument, possibly with Godfrey's active assistance. This might not have been a good testimonial to the instrument's accuracy, or a previous error may account for the faulty charting, but the location of this headland shows an error of ten miles as judged by the latest survey. However, Fisher pronounced his approbation of the instrument.

Having secured Fisher's approval, the next step was to have a sea test. Steward took the instrument with him when the sloop *Trueman* sailed from Philadelphia on November 28th on a cruise to the West Indies. He and Captain Cox made numerous observations of the sun, moon, and stars. That it fulfilled all expectations was attested by Steward and Cox in their affidavit.

In their efforts to establish Godfrey as the sole inventor of the reflecting quadrant some of his over-zealous supporters have sought to imply that on this voyage, Steward met John Hadley's nephew and showed him the instrument.[5] The inference is that Hadley constructed his instrument from a description of Godfrey's furnished by his nephew. There does not appear to be the slightest justification for any such belief. Certainly, Hadley's nephew was in Jamaica at the time, but it is extremely improbable that information concerning Godfrey's quadrant could have reached him in time for Hadley to work out the mathematical basis and to prepare the paper which he read before the members of the Royal Society on May 3, 1731. Moreover, Hadley described his quadrant as more useful to surveyors for measuring angles, but expressed a doubt that it might be applicable to navigation. Finally, he was fully qualified to design the reflecting quadrant without any assistance.

Now that the value of his instrument had been established by these practical tests, Godfrey wanted to draw some financial benefit from it. This was no easy matter in those days of slow communications. There

[4] Published in London, 1756.

[5] J. F. Watson, "Annals of Philadelphia," Philadelphia, Edwin S. Stuart, 1884, Vol. 1, p. 529; and *Pennsylvania Magazine* (Historical Society of Pennsylvania), Vol. 58, p. 291 (1927).

were no journals in America with a distribution beyond the immediate localities in which they were printed. Nor were there as yet any learned societies in the country which could put the stamp of their approval upon the instrument. Godfrey was ambitious, he wanted to have his quadrant known in London, capital of the greatest maritime nation, where the market would be largest. With this hope in mind, he left one of his quadrants with Logan on the understanding he would undertake to secure the interest of his influential friends in promoting it, and to notify the members of the Royal Society of its design, construction, and application.

Doubtless, Logan had the best of intentions but, being a busy public official with heavy responsibilities, he was too fully occupied to give the matter his attention. Godfrey may have had to remind him that he was impatient to receive the rewards of his invention. In any event, Logan neglected to act for a full twelve-month. Finally, he addressed a long letter to Dr. Edmund Halley, the astronomer royal and secretary to the Royal Society, describing Godfrey's instrument and requesting that the letter be communicated to the Royal Society.

For some inexplicable reason Halley did not communicate the letter to the Society. The omission cannot be attributed to a lack of interest in navigation because on November 16, 1692, he had described to the members of the Society "A Sea Quadrant wherein both the Horizon and the Object shall be seen distinctly and enlarged at one view in the common focus of the Telescope." However, he admitted that this instrument had been forestalled by one made by Robert Hooke, which had been ordered by the Society to be tried at sea, although there is no record of any test having been conducted. It is not unreasonable to suppose that Halley also knew that Newton had designed a similar instrument as early as 1677, which had come into the possession of Thomas Heath, a famous instrument maker, who had exhibited it in his shop on the Strand, next to the Fountain Tavern.[6]

The explanation for Halley's inaction can most probably be found in the fact that John Hadley had only recently submitted two models of octants, and had described their operation to the Royal Society.[7] It can only be assumed that when Halley compared the newly arrived account of Godfrey's mariner's bow with Hadley's octant, which had been in his possession for nearly a year, he arrived at the conclusion the two instruments were identical and he could see no reason for giving publicity to one that was brought to the attention of the Society which was already acquainted with the other. Logan's letter was laid aside.

Since Halley did not acknowledge receipt of Logan's letter, or report any action upon it, and a description of Hadley's octant had reached

[6] E. G. R. Taylor, "The Mathematical Practitioners of Tudor and Stuart England," New York, Cambridge University Press, 1954, p. 305.

[7] *Philosophical Transactions* of the Royal Society, Vol. VII, p. 669–673 (1733–34).

Philadelphia, Logan wrote a second letter, dated December 4, 1733. This was addressed to Peter Collinson with the request he would report it to the Royal Society. Collinson was a wealthy Quaker merchant, an amateur scientist, prominent in the Royal Society, and numbered many Americans among his correspondents, including Benjamin Franklin. He was noteworthy for stimulating the interest of several Americans in science, but his greatest contribution was in the trouble he took to ensure that their efforts were recognized and used.

In this letter which Collinson promptly communicated to the Society[8] Logan appears to have realized the harm done to Godfrey's claim to priority by the delay in advancing it. He enclosed copies of the affidavits we have mentioned and described the chronology of the invention. He conceded there was a resemblance between one of Hadley's instruments and that of Godfrey, but expressed the hope the members of the Royal Society would express their approval of the latter instrument so that it might be better known to the maritime world. "Some masters of vessels who sail from hence to the West Indies have them made as well as they can be done here, and have found so great an advantage in the facility and in the ready use of them in these southerly latitudes, that they reject all others. And it can scarcely be doubted, but when the instrument becomes more generally known it may, on the Royal Society's approbation, if it appear worthy of it, more universally obtain in practice." [9]

Unfortunately the presentation of these documents did not evoke any expression of approval that could gratify Godfrey's wish to promote public interest in his instrument.

This failure to obtain recognition provoked Logan into writing one more letter to the Royal Society, on June 28, 1734. In this he was principally concerned with establishing Godfrey's priority of invention over Hadley, in spite of the delay in entering the claim. Logan lays emphasis on Godfrey's early work, pointing out that the instrument mentioned in the affidavits he had submitted was not the first model but one which included improvements and refinements upon the first. He writes: "It is now four years since Tho. Godfrey hit upon this improvement; for his account of it, laid before the Society last winter, in which he mentions two years, was written in 1732. And in the same year, 1730, after he was satisfied in this, he applied himself to think of the other, viz. the reflecting instruments by speculums for a help in the case of longitude [? latitude] though it is also useful in taking altitudes, and one of these, as has been abundantly proved by the maker, and those who had it with them, was taken to sea, and there used in taking the latitude, the winter of that year and brought back again hither before the end of February, 1730–31."

[8] Cf. footnote 7, pp. 669–673.
[9] Cf. footnote 7, p. 672.

This leaves us in doubt upon the precise date of both Hadley's and Godfrey's instruments. Logan says he learned of Godfrey's instrument in June, 1730. The only date advanced for Hadley's invention places it in the summer of the same year, which would give the American precedence by a few weeks. Irrespective of the dates, Hadley has received all the credit for the invention. Indeed poor Godfrey does not appear to have gained any recogniton even among his fellow townsmen for, thirty-two years after his death, James Ham was making and selling nautical instruments "at the sign of Hadley's Quadrant" below the Drawbridge in Front and Water Streets in Philadelphia.

For a man who had the support of the influential Logan this neglect is difficult to comprehend. The only possible explanation may be found in the legend which grew up around Godfrey's reputation and which suggests his later life was marked by excessive drinking.

The legend asserts that the Royal Society agreed that no distinction could be drawn between Godfrey and Hadley, their merits were equal, the development of their instruments was practically simultaneous and independent. Where there was so much similarity, the Society announced the award of a prize of £1000 to each of the inventors. When this became known, it is alleged, Godfrey's best friends persuaded the Society's officials to refrain from paying him in cash, as this would speedily be dissipated on strong spirits. Instead, they suggested the money should be invested in household furniture.

Unfortunately for the legend there is no record of any such reward having been granted by the Society, moreover, it is extremely improbable the members would have shown any moral discrimination in Godfrey's disfavor had such an award been made.

Finally, it seems unjust to accuse Godfrey of over-indulgence in liquor. Had he been the alcoholic he is pictured as being we might have expected Franklin to have alluded to it in his mention of Godfrey's other character defects. But had he been guilty of intemperance it is improbable that Logan would have encouraged intimacy with him. The strictly formal Quaker could hardly be expected to display a public interest in an inebriate and reprobate. What could be the possible origin of the legend is that the inventor, having placed high hopes upon making money out of his quadrant, suffered grave disappointment over the failure to do this, and that he did what so many others have done before and since, sought consolation in drink. That he could have been assisted in this through gifts of money, mistakenly supposed to have come from the Royal Society, should not be excluded from consideration. In those days of unorganized markets and limited commercial facilities, it was not an uncommon practice for wealthy men to make gifts of money to those whom they thought worthy of encouragement. Whatever individual members may have given him, there are no grounds for supposing the Royal Society made any grant, either in cash

or in kind. Nor should it be said there is any evidence outside the legend that Godfrey was a drunkard.

As opposed to this bad reputation given him there is some evidence that he enjoyed a certain amount of acceptance among his fellow citizens. In addition to his election to membership of the American Philosophical Society and the Library Company, he appears among the promoters of the first hospital (Pennsylvania Hospital, founded by Benjamin Franklin and Dr. Bond) and the Philadelphia Contributionship, the first fire insurance company. All these institutions were prominently associated with Benjamin Franklin.

Godfrey inherited his father's estate when he came of age in 1725 but he sold the farm ten years later; payment for this may account for his sudden access to funds which an ignorant public attributed to the gift of the Royal Society.

He died, apparently in reduced circumstances, in 1749 at the early age of forty-five. His sole memorial is a stone erected by public subscription in Laurel Hill cemetery in Philadelphia in 1843, marking the grave to which his remains were removed from the family burial ground on the old farm in Bristol Township.

Research in Metal Sputtering Shows Printed Circuit Possibilities.—Recent research in the field of cathode metal sputtering at Bell Telephone Laboratories indicates that this century-old technique may be useful in producing precision printed circuits for modern communication equipment. It now appears that entire circuits, including resistors, capacitors, and leads, may be laid down by this technique, in which ionized gas molecules bombard a cathode, dislodging atoms of metal which then redeposit on nearby surfaces.

Harold Basseches of Bell Laboratories has produced thin films of a number of electrically interesting high melting point metals. Tantalum and titanium, for example, melting at 3000°C. and 1670°C. respectively, can be laid down in films which show sufficiently high resistivity to be useful as resistors in printed circuits. With proper masking of the substrate, lines and patterns of practically any desired shape and size can be formed, ranging in width down to a few mils. These sputtered films generally are between a few hundred and a few thousand angstroms thick.

In addition to pure metals, alloys such as those of nickel-copper and nickel-chromium can be sputtered without difficulty, apparently retaining their approximate original composition.

R. W. Berry, also of Bell Laboratories, has produced "printed capacitors" by a combination of sputtering and chemical methods. A tantalum film of the proper shape and size is first sputtered onto the substrate and then anodically oxidized to form a tantalum oxide dielectric film. The counter electrode, a film of gold, can then be evaporated onto the dielectric to form the completed capacitor "sandwich."

Copper leads can be sputtered without difficulty to connect the various components. The technique is attractive since it eliminates the need of any organic adhesives.

While some disagreement still exists as to the exact mechanism of cathode sputtering, its effects can be easily described and recognized. These effects were first noted in 1852 by W. R. Grove, and in 1858 by J. Plücker.

In cathode sputtering, a plate of the metal to be deposited is used as a cathode. The substrate on which the film is to be deposited is placed on a table close to the cathode. After evacuation, argon or other suitable gas is introduced and maintained at a pressure of approximately 20 to 40 microns.

When a voltage is applied, ionized atoms of the gas bombard the cathode, dislodging metal atoms or clusters of atoms, which then deposit on the substrate.

In "reactive sputtering," films of inorganic compounds are formed by introducing a small controlled amount of a reactive gas such as oxygen, nitrogen, or hydrogen sulfide into the apparatus. Compounds which can be formed in this way include the oxides, nitrides, and sulfides of a number of metals.

Sputtering is one of the most convenient methods available for the production of thin films of high melting point metals. In general, films produced by sputtering are strongly adherent. Another inherent advantage of sputtering is that the thickness of films so produced may be controlled within narrow tolerances.

AUTOMATIC MINIMUM WEIGHT DESIGN OF STEEL FRAMES*

BY

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SUMMARY

The automatic plastic design of structural frames can be treated by the method of linear programming. The number of variables, however, increases so fast with the complexity of the frame that only simple frames can be handled by this method even on a large electronic computer. In the present paper a method is proposed, which considers alternately two different requirements that a frame must satisfy, and thereby greatly reduces the size of the problem.

Following the Introduction, Part I of the paper presents the method with reference to a simple numerical example; Part II establishes the general applicability of the proposed method. Part III presents some lemmas of practical importance, and some discussion, with examples, of special considerations that may arise in the design of actual frames.

INTRODUCTION

The type of frame to be discussed in this paper consists of horizontal beams and vertical columns, each joint being rigid enough to transmit the full plastic moment of any member framing into it. Between joints, any member will be assumed to have uniform cross-section. This type of frame forms the essential unit of many types of practical construction, and the plastic or limit *analysis* of such frames has been developed (1, 2, 3).[3]

The following problem will be discussed in this paper: Given the center-line dimensions of a frame, and a system of loads acting on it, how should the cross-sections of the members be assigned so that the material consumption is a minimum?

This problem will be labelled plastic *design* of frames, to distinguish it from the problem of plastic *analysis* which has been discussed almost exclusively in the literature. It should be noted, however, that the terms plastic analysis and plastic design are often used promiscuously. In fact, the design method of the present paper can be made subject to various overriding restrictions; one of these could be that the full plastic moments of all members in a frame were equal, or bore given ratios one to another, and this restricted problem would then be equivalent to that of analysis, since only one variable would have to be de-

* The method presented in this paper was developed in the course of research sponsored by the International Business Machines Corporation of New York City.

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[3] The boldface numbers in parentheses refer to the references appended to this paper.

termined. The present minimum weight design method, therefore, includes analysis as a special case.

The importance of a minimum weight solution to a structural problem lies in the fact that the designer is given an absolute basis with which to compare any other design. Should a practical design depart from the minimum, he can calculate the resulting penalty in material consumption and estimate the penalty in cost. Such departures may be dictated by architectural considerations; other important modifications may be dictated by the consideration of the stability of the columns, which does not normally enter into plastic calculations. However, it will be shown that a restriction on column size can be incorporated into the present method.

With the normal assumptions made in plastic analysis of frames, it may be shown that the bending moment distribution at collapse must satisfy three conditions, which will be referred to as the conditions of equilibrium, mechanism, and yield. The condition of equilibrium states that the bending moment distribution must be in equilibrium with the applied loads, and that of mechanism requires that the bending moment reaches the full plastic moment at a sufficient number of cross-sections ("hinges") so that the frame becomes a mechanism of one (or more) degrees of freedom. The signs of the bending moments at the hinges correspond to the signs of the hinge rotations in a possible infinitesimal motion of the mechanism. Finally, the condition of yield states that the absolute value of the bending moment shall nowhere exceed the full plastic moment. For problems of analysis, it is possible to construct "safe" or upper bound solutions for the full plastic moments by considering only the conditions of equilibrium and yield; this is the basis, for example, of Horne's (4) method of plastic moment distribution. Similarly, by considering only the conditions of equilibrium and mechanism, it is possible to construct lower bound ("unsafe") solutions; this is the basis of Neal and Symonds' (5) method of combination of mechanisms.

Similar approaches to a solution are possible for the design problem. Heyman (6) gave a formal solution making use of the equilibrium and yield conditions, in which the problem was set up in terms of continued linear inequalities; at any stage in the process, a design is obtained which furnishes an upper bound on the minimum weight. Similarly, in the course of an extensive discussion, Foulkes (7) proposed a solution in terms of mechanisms which furnishes lower bounds on the minimum weight. Each of these methods can be shown to be equivalent to a problem in linear programming; the number of variables and restricting inequalities is so large, however, even for a fairly simple frame, that it is not possible to obtain practical solutions in these terms on existing computers. Livesley (8) programmed the problem for EDSAC at Cambridge University, but obtained a solution for, in effect, a frame of

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only four storeys and a single bay. The largest frame for which a solution has been obtained manually appears to be one of two bays and two storeys (2).

This discrepancy between the structural complexity of a frame and the size of the corresponding linear programming problem stems from the fact that of the very large number of restraining inequalities on the variables, most prove to be inoperative when the final solution is obtained. However, there is no method of ruling out inequalities *a priori*. Considering an upper bound calculation (based on equilibrium and yield), a designer would in fact bear in mind that the final solution must correspond to a mechanism, and if he has an idea of where some, at least, of the hinges will form, his calculation can be correspondingly shortened by using equations instead of inequalities at the guessed hinge positions. The dual calculation (Foulkes' method) can also be shortened considerably by rejecting *a priori* those mechanisms which are unlikely to occur, but this process of rejection requires an inspired choice of which a machine is not capable, and in which even the most gifted designer is likely to err as the complexity of the structure increases.

The method of solution presented in this paper alternately uses the conditions of equilibrium and yield (Step I) and the conditions of equilibrium and mechanism (Step II). By considering, in this way, *all* conditions to which the frame is subject, the number of variables that have to be considered at any one stage is greatly reduced. Nevertheless the steps in the solution are automatic in the sense that no choice is available to the designer for the progressive modifications to be made in the solution; each move in Step I reduces the structural weight by a finite amount, and Step II indicates whether the final solution has been obtained, and, if not, what adjustments should be made in order to move into a position where Step I can again be applied. In this sense the proposed method is suitable for programming on an electronic computer.

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PART I

The following analysis is based on the simplifying assumption that the designer has at his disposal a continuous spectrum of structural cross-sections rather than the discrete sizes that are commercially available. Only concentrated loads will be considered, because the effect of distributed loads can be produced with any desired degree of approximation by suitably chosen concentrated loads. Under concentrated loads extreme absolute values of the bending moment in a certain member can only occur at its ends and the points of application of the loads carried by this member. These *potentially critical sections* will be numbered throughout the frame; lower case Latin subscripts will be used to refer to generic critical sections.

Several members may be required to have the same full plastic moment. It is then convenient to speak of these members as forming

a *span*, and to use the term *length of span* for the sum of their lengths. This terminology will be used consistently, even if a span contains only a single member. Capital Latin subscripts will be used to refer to the spans.

As is customary, a linearized weight function will be used to estimate the structural weight. If l_K is the length of the span with the full plastic moment M_K, the total weight of the frame is supposed to be proportional to

$$W = \sum M_K l_K, \tag{1}$$

where the summation includes all spans. It is well known that this linearization does not severely restrict the value of the results obtained by the use of the expression (1) (see, for instance, (9)). For brevity, W will be called the *weight* in the following.

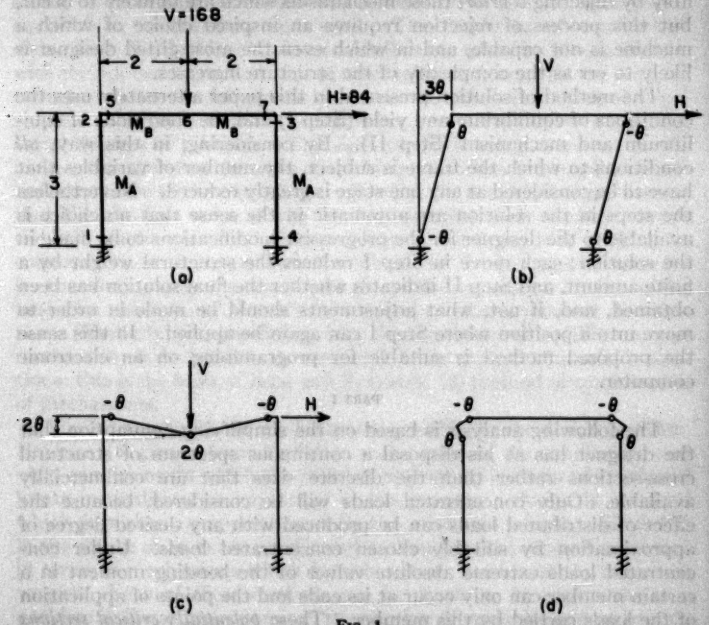

FIG. 1.

Consider the minimum weight design of the simple frame in Fig. 1*a*, when the columns are to have the same full plastic moment M_A, possibly different from the full plastic moment M_B of the beam. Note that it is

not necessary to specify the units of force and length for the purposes of calculation, so long as it is understood that the full plastic moments obtained from the calculation are expressed in the same units of force and length. There are seven potentially critical sections (labelled 1 through 7 in Fig. 1*a*). A bending moment distribution is specified by the values of the bending moments M_1 through M_7 at these sections. These bending moments, however, cannot be assigned arbitrarily; as the frame has three redundancies there must be four independent equations of equilibrium connecting the bending moments M_1 through M_7.

The simplest way of obtaining four linearly independent equations of this kind is to apply the principal of virtual work to four linearly independent mechanisms. Consider, for instance, the sidesway mechanism of Fig. 1*b* which involves hinge rotations of the same absolute value θ at sections 1, 2, 3, and 4. A bending moment will be considered positive if it produces tensile stresses on the inside of the frame, and a hinge rotation will be considered positive if it involves an increase in angle when judged from the inside of the frame. With this sign convention, application of the principle of virtual work to the mechanism in Fig. 1*b* furnishes the following equation of equilibrium:

$$-M_1 + M_2 - M_3 + M_4 = 252. \tag{2}$$

Similarly, the beam mechanism of Fig. 1*c* yields

$$-M_5 + 2M_6 - M_7 = 336. \tag{3}$$

The remaining two equations of equilibrium correspond to the two joint rotations that are shown together in Fig. 1*d*:

$$M_2 - M_5 = 0 \tag{4}$$

$$M_3 - M_7 = 0. \tag{5}$$

The coefficients and right-hand sides of Eqs. 2 through 5 appear in lines 1 through 4 of Table I. These four lines contain the complete data required for the solution of the problem. Line 5 labelled *Step 0* contains a set of bending moments M_1 through M_7 that satisfy the four equations of equilibrium. Note that there are infinitely many equilibrium distributions of bending moments; any one of these may be used to start the computation. The largest absolute values of bending moment are $|M_1| = 252$ in span A and $M_6 = 188$ in span B. The design with the fully plastic moments

$$M_A = 252, \quad M_B = 188 \tag{6}$$

will therefore certainly carry the given loads; its weight

$$W = 6M_A + 4M_B = 2264 \tag{7}$$

TABLE I.

Step	Line	Plastic Moment Section		M_A 1	2	3	4	M_B 5	6	7	
Data	1			-1	1	-1	1				252
	2							-1	2	-1	336
	3				1			-1			0
	4					1				-1	0
0	5			-252*	30	10	-20	30	188*	10	W=2264
I	6	6×136=816	m_A	1			1				
	7	4×66=264	m_B			-2	-2		-1	-2	
	8		x		1	-1	-2	1		-1	
	9	$m_A=136$		136			136				
	10			-116	30	10	116*	30	188	10	W=1448
	11	6×43=258	m_A	1	1	-1	-1	1		-1	
	12	4×99=396	m_B		-1	-1		-1	-1	-1	
	13	$m_B=99$			-99	-99		-99	-99	-99	
	14			-116	-69	-89	116	-69	89	-89*	W=1052
	15	8×9=72	m_B	-2	-3	1	2	-3	-1	1	
	16	$m_B=-9$		18	27	-9	-18	27	9	-9	
	17			-98	-42	-98*	98	-42	98	-98	W=980
II	18			-		-	+		+	-	
	19	α	1+2-3+4	-1			1		2	-2	
	20	β	4			1				-1	
	21			-(α)		-(-β)	(α)		(2α)	-(2α+β)	
	22	$\alpha=5/3, \beta=-8/3$		-(5/3)		-(8/3)	(5/3)		(10/3)	-(2/3)	
I	23	$m_A=43$		43	43	-43	-43	43		-43	
	24			-73	73*	-33	73	73	188	-33	W=1190
	25	14×10=140	m_A	1	-1	-3	-1	-1	-2	-3	
	26	$m_A=10$		10	-10	-30	-10	-10	-20	-30	
	27			-63	63	-63*	63	63	168	-63	W=1050
II	28			-	+	-	+		+		
	29	α	1	-1	1	-1	1				
	30	β	2-3-4		-1	-1			2		
	31			-(α)	(α-β)	-(α+β)	(α)		(2β)		
	32	$\alpha=3/2\ \beta=2$		-(3/2)	(-1/2)	-(7/2)	(3/2)		(4)		
I	33	2×35=70	m_A	1	3	1	-1	3	2	1	
	34	$m_A=-35$		-35	-105	-35	35	-105	-70	-35	
	35			-98	-42	-98	98	-42	98	-98*	W=980

is an upper bound for the minimum weight of a frame capable of carrying these loads.

For a given distribution of bending moments, any section i at which the bending moment M_i equals the fully plastic moment M_K of the span will be called a *hinge*. The terms *positive hinge* or *negative hinge* will be used to indicate that $M_i = M_K$ or $M_i = -M_K$. In general, each span will only contain one hinge at the end of Step 0. In line 5 of Table I the hinge locations have been marked by asterisks.

In Step I, the bending moment distribution of Step 0 (line 5 of Table I) will be modified so as to maintain equilibrium and preserve the hinges obtained in the preceding step, while at the same time the weight is reduced. The transition from one equilibrium distribution of bending moments to another is effected by a set of residual moments m_i, which correspond to a state of self-stress in the frame. In the present example, the residual moments m_1 through m_7 must satisfy the homogeneous forms of Eqs. 2 through 5:

$$\begin{aligned} -m_1 + m_2 - m_3 + m_4 &= 0 \\ -m_5 + 2m_6 - m_7 &= 0 \\ m_2 - m_5 &= 0 \\ m_3 - m_7 &= 0. \end{aligned} \tag{8}$$

As a consequence of this, there cannot be more than three independent residual moments. These will be chosen to reduce the absolute values of the bending moments at the hinges and thereby M_A and M_B. Let m_A and m_B denote the reductions in M_A and M_B, respectively. These reductions are effected by setting $m_1 = m_A$ at the negative hinge 1 and $m_6 = -m_B$ at the positive hinge 6. These two residual moments do not yet specify the most general distribution of residual moments; the choice of m_1 and m_6 must therefore be supplemented by, say $m_2 = x$. For convenient reference m_A and m_B will be called *attacking moments* and x a *supplementary moment*. With

$$m_1 = m_A, \quad m_2 = x, \quad m_6 = -m_B \tag{9}$$

Eqs. 8 furnish

$$\begin{aligned} m_3 &= -2m_B - x \\ m_4 &= m_A - 2m_B - 2x \\ m_5 &= x \\ m_7 &= -2m_B - x. \end{aligned} \tag{10}$$

The coefficients of m_A, m_B, and x on the right-hand sides of (9) and (10) are displayed in lines 6, 7, and 8 of Table I. From these, the most general distribution of residual moments can be obtained by linear combination.

Consider first the reduction in structural weight that can be obtained by the use of line 6, which yields the following bending moments in span A:

$$M_1 = -252 + m_A, \quad M_2 = 30, \quad M_3 = 10, \quad M_4 = -20 + m_A. \tag{11}$$

Since the full plastic moment M_A must equal the largest absolute value in (11), the greatest reduction in M_A is achieved by so choosing m_A that $M_1 = -M_4$, that is, $m_A = 136$. Line 9 shows the corresponding residual moments; their superposition furnishes the bending moments in line 10. The position of the new hinge (section 4) has been marked by an asterisk in line 10. The reduction in weight obtained by the use of line 6 is $6 \times 136 = 816$; it has been entered at the beginning of line 6. The reduced weight, $2264 - 816 = 1448$ has been entered at the end of line 10.

If line 7 had been used instead of line 6, a new hinge would have appeared at section 7 for $m_B = 66$. (Note that this value of m_B does not necessitate a change in M_A.) The corresponding saving in the structural weight, $4 \times 66 = 264$, is shown at the beginning of line 7; since it is smaller than the saving shown in line 6, use of line 6 is more efficient.

Since the suppression of a hinge will only be allowed in Step II, any further modification of bending moments in Step I must maintain the new hinge at section 4 together with the previous hinges at sections 1 and 6. Since the hinge at 1 is negative, and the hinges at 4 and 6 are positive, the residual moments at these hinges must be chosen as follows:

$$m_1 = m_A, \quad m_4 = -m_A, \quad m_6 = -m_B. \tag{12}$$

From lines 6 through 8, the second of these conditions furnishes $m_A - 2m_B - 2x = -m_A$ or

$$x = m_A - m_B. \tag{13}$$

With this value of x, lines 6 through 8 furnish

$$\begin{aligned} m_2 &= m_A - m_B \\ m_3 &= -m_A - m_B \\ m_5 &= m_A - m_B \\ m_7 &= -m_A - m_B. \end{aligned} \tag{14}$$

The coefficients of m_A and m_B on the right-hand sides of (12) and (14) are given in lines 11 and 12. Note that the appearance of a new hinge has reduced by one the number of independent parameters in the most general distribution of residual moments that can be used to achieve a further reduction in weight.

By the same procedure as before, it is found that a saving in weight of 258 is achieved by the use of line 11 and a saving of 396 by the use of line 12. Accordingly, line 12 is used with $m_B = 99$. The corresponding residual moments are shown in line 13; their addition to the bending moments of line 10 furnishes those in line 14. The asterisk in this line marks the new hinge at section 7.

The positive hinges at 4 and 6, and the negative hinges at 1 and 7 require that

$$m_1 = m_A, \quad m_4 = -m_A, \quad m_6 = -m_B, \quad m_7 = m_B \tag{15}$$

for the continuation of Step I. Note that lines 11 and 12 already take care of the first three of these equations, and that the last Eq. 15 furnishes $-m_A - m_B = m_B$ or

$$m_A = -2m_B. \tag{16}$$

With this value of m_A lines 11 and 12 furnish the residual moments displayed in line 15. Note that the appearance of a new hinge has again reduced by one the number of independent parameters in the most general distribution of residual moments that can be used to achieve a further saving in weight. Whereas previously the changes in M_A and M_B could be discussed separately, they are now coupled and must be considered jointly. As $m_B = 1$ produces a weight decrease of 4×1 in span B but an increase of $6 \times 2 = 12$ in span A, the attacking moment m_B must be given negative values to achieve a decrease in weight. As before it is found that $m_B = -9$ for greatest reduction in structural weight. Line 16 shows the corresponding residual moments and line 17 the bending moments obtained by their superposition on those in line 14.

A new hinge is seen to form at section 3. As it is obviously impossible to superimpose further residual moments on the bending moment distribution in line 17 in such a manner that the hinges at sections 1, 3, 4, 6, and 7 are maintained, Step I has been completed, and Step II must now be used.

In the present example, the supplementary moment x disappeared as an independent parameter before the number of independent attacking moments was reduced during Step I. In general, there is a set of supplementary moments $x_1, x_2, \cdots$, and a set of attacking moments $m_A, m_B, \cdots$, and it is by no means necessary that all supplementary moments have disappeared as independent parameters before the number of independent attacking moments can be reduced. Step I is completed, when it is no longer possible to reduce the structural weight by an appropriate choice of attacking moments.

The preceding description of Step I was couched in purely static terms. The following kinematic considerations will however be useful

in the later discussion. Step 0 indicated hinges at sections 1 and 6. To yield a mechanism, these hinges must be supplemented by two further hinges. One of these will be taken at section 2 in accordance with the choice of $m_2 = x$ as supplementary moment (see Eq. 9); the second additional hinge will be taken at whatever section it is desired to find the residual moment in terms of m_A, m_B, and x. In Fig. 2*a*, for instance, this hinge has been taken at section 4. With the bending moments of Eq. 9, the principle of virtual work applied to the mechanism of Fig. 2*a* furnishes the second Eq. 10. (Note that $H = V = 0$ for this discussion of *residual* moments.) The other Eqs. 10 can be similarly obtained by taking the fourth hinge at one of the sections 3, 5, or 7 instead of the section 4.

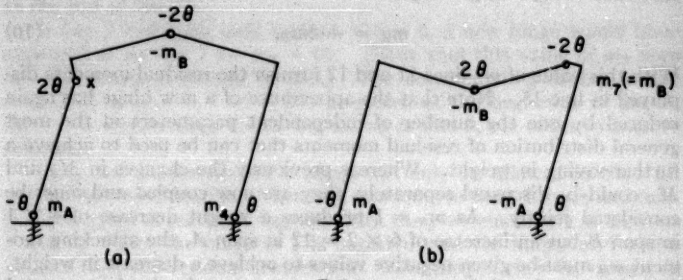

FIG. 2.

Line 10 of Table I indicates a new hinge at section 4. As above, the condition that this new hinge be maintained during the continuation of Step I requires that $m_4 = -m_A$. With this value of m_4, the mechanism of Fig. 2*a* furnishes Eq. 13, which deletes $x(= m_2)$ from the list of independent parameters. Accordingly, the hinge at section 2 should now be suppressed, and mechanisms should be considered that have hinges at 1, 2, 4, and some fourth section. In Fig. 2*b*, for instance, the fourth hinge is chosen at section 7. With the bending moments in Eq. 12, this mechanism yields the last Eq. 14, and the other Eqs. 14 can be obtained in a similar manner.

Line 14 indicates the formation of a negative hinge at section 7. Since this hinge is to be maintained during the continuation of Step I, $m_7 = m_B$. With this value of m_7, the mechanism of Fig. 2*b* furnishes Eq. 16, which suppresses say m_A as an independent parameter. Whereas the disappearance of the *supplementary* moment $x(= m_2)$ as an independent parameter was reflected in the elimination of the hinge at section 2, the deletion of the *attacking* moment m_A from the list of independent parameters cannot be so reflected. Thus, on the next

operation, the formation of a new hinge at section 3 (line 17) introduces an additional degree of freedom, and a generic displacement of the resulting mechanism of two degrees of freedom can be obtained by the linear combination of the displacements shown in Fig. 2*b* and the right-hand half of Fig. 1*d*.

To maintain the new hinge at section 3, the residual moments there would have to be given the value $m_3 = m_A$; according to Eq. 16 this means that $m_3 = -2m_B$. Since $m_7 = m_B$, the equation of equilibrium in line 4 of Table I then furnishes $m_B = 0$, and Eq. 16 yields $m_A = 0$. Thus, Step I cannot be continued any further and Step II must be initiated.

Line 18 of Table I indicates the locations and signs of the hinges of the bending moment distribution in line 17, which corresponds to the design with $M_A = M_B = 98$ and $W = 980$. The insertion of hinges at these five sections certainly transforms the frame into a mechanism of two degrees of freedom, though this mechanism would not necessarily be capable of an infinitesimal displacement in which the hinge rotations have the signs deduced from the bending moments in line 17.

Now Foulkes (7) has shown that for the minimum weight design there exists a collapse mechanism for which the sum of the absolute values of the hinge rotations in each span is proportional to the length of this span. In Step II, this criterion is used to test whether a minimum weight design has been achieved in Step I; if not, Step II indicates which hinge must be eliminated to prepare the way for another application of Step I.

The data equations in line 1 though 4 of Table I were obtained from the four independent mechanisms shown in Figs. 1*b* through 1*d*, and any other mechanism may be obtained from these by linear combination. The coefficients on the left-hand sides of the data equations may be interpreted as the hinge rotations of the four basic mechanisms. The mechanism of line 18 does not possess any hinges at sections 2 and 5. Accordingly, the data equations must be combined to furnish zero coefficients in columns 2 and 5. It is seen that line 4 does already fulfill this condition, and so does the combination $1 + 2 - 3 + 4$ of the data equations, which corresponds to the mechanism of Fig. 2*b*. Lines 19 and 20 represent these independent mechanisms that do not have hinges at sections 2 and 5. Their most general linear combination is given in line 21, in which all quantities enclosed in parentheses must be positive if the hinge rotations are to have the signs indicated in line 18. Foulkes' criterion therefore requires that

$$\begin{aligned} 2\alpha - \beta &= 6, \\ 4\alpha + \beta &= 4, \end{aligned} \tag{17}$$

where the coefficient of proportionality has been omitted because the

hinge rotations of a mechanism are only defined to within an arbitrary common factor.

Substitution of the solution of (17) into line 21 furnishes the hinge rotations of a mechanism that formally satisfies the Foulkes criterion: in each span, the sum of the quantities enclosed in parentheses equals the length of the span. In other words, the absolute values of the hinge rotations in each span can be combined with factors ± 1 to furnish the length of the span. For a "true Foulkes mechanism" each of these factors must be $+1$. The term "false Foulkes mechanism" will be used when at least one of the factors has the value -1. Hinges at which the factor -1 must be used will be called "hinges of the wrong sign."

Equations 17 yield $\alpha = 5/3$ and $\beta = -8/3$ and substitution of these values into line 21 furnishes the hinge rotations displayed in line 22. Since each parenthesis has a positive content, a true Foulkes mechanism has been obtained. The values $M_A = M_B = 98$ from line 17 therefore represent a minimum weight design.

It will often be found that Step II leads to a false Foulkes mechanism. To illustrate this, assume that a different choice is made at line 13, using $m_A = 43$ instead of $m_B = 99$. Lines 23 through 27 show the computations up to the end of Step I, and lines 28 through 32 show the subsequent Step II. In line 32, the second parenthesis has a negative content. A false Foulkes mechanism has therefore been obtained with a hinge of the wrong sign at section 2, and the minimum weight design has not yet been achieved.

A repetition of Step I is now made possible by dropping the requirement that the bending moment distribution in line 27 should be modified so as to maintain the hinge at section 2. The ensuing Step I (lines 33, 34, and 35) leads to the bending moments of line 35, which are identical with those in line 17. The subsequent Step II would therefore be represented by lines 18 through 22; it would show that a minimum weight design has been obtained.

In this example, there appeared only one hinge of the wrong sign. A false Foulkes mechanism may however possess several hinges of wrong sign. Step I can then usually be repeated after only one of these hinges is dropped. There are exceptional circumstances, however, in which several hinges may have to be dropped simultaneously before Step I can be repeated, as will be discussed later. (A numerical example is given in Part III.)

PART II

The general validity of the method illustrated in Part I will now be discussed. Consider a frame that is to carry a given set of concentrated loads. Let this frame have r redundancies, s potentially critical sections, and $N < s - r$ spans, the full plastic moments of which are to be chosen to minimize the structural weight.

The computing form (Table I) for this frame starts with $s - r$ lines of data, which may be derived by applying the principle of virtual work to $s - r$ independent mechanisms. The data may be interpreted as furnishing: (*i*) a set of independent non-homogeneous equations of equilibrium for the bending moments M_i at the critical sections when the frame carries the given loads, (*ii*) a set of independent homogeneous equations of equilibrium for the residual moments m_i of a state of self-stress, or (*iii*) the hinge rotations of a set of independent mechanisms from which all other possible mechanisms may be obtained by linear combination.

Step 0

In Step 0 a system of bending moments M_i at the critical sections is chosen that satisfies the non-homogeneous data equations. The largest value of $|M_i|$ in each span is then taken as the full plastic moment M_K of this span. A section i at which $|M_i|$ equals the full plastic moment M_K of the span is called a positive or negative hinge according to whether $M_i = M_K$ or $M_i = -M_K$. Step 0 produces at least one hinge in each span.

Any Step I is preceded by either Step 0 or a Step II. In either case, the preceding step indicates $h \geq N$ hinges that should be maintained during the ensuing Step I; it also furnishes bending moments M_i and full plastic moments M_K such that $|M_i|$ equals the full plastic moment of the span at each indicated hinge and does not exceed the full plastic moment of the span at any other critical section.

Step I

The most general equilibrium distribution of residual moments m_i is now constructed that will maintain the indicated hinges when it is superimposed on the bending moments M_i.

In the actual computation, these residual moments are expressed in terms of a number of independent parameters, which will in general include some attacking moments and some supplementary moments. The condition for the maintenance of each newly formed hinge is then used to eliminate one of the hitherto independent parameters from the subsequent computation. If possible, one of the supplementary moments is eliminated in this way. If, however, the condition for the maintenance of the newly formed hinge does not contain any supplementary moment, it establishes a relation between attacking moments and is used to eliminate an attacking moment.

For the purposes of the following discussion it is preferable to eliminate supplementary moments as before but to abstain from actually eliminating attacking moments; as a consequence of this, the full original array of attacking moments is subject to an increasing number of

linear relations. Each of these relations is essentially an equilibrium condition for residual moments which have been set equal in absolute value to the attacking moments of their spans. As is illustrated by the example in Part I, each such relation between attacking moments corresponds to a mechanism. Equation 16, for instance, may be obtained by applying the principle of virtual work to the mechanism of Fig. 2*b* after stipulating that m_7 equal m_B. It is of course unnecessary to draw the mechanism to derive Eq. 16; the relative rotations in the hinges of this mechanism may be obtained by combining the homogeneous data equations to suppress the rotations at sections 2, 3, and 5, at which no hinges have been formed.

As a rule, each phase of Step I involves the formation of *one* additional hinge. Since each of the newly formed hinges is maintained until the end of the considered Step I, the relations between attacking moments are in general derived from a set of independent mechanisms and constitute, therefore, a set of independent homogeneous linear equations. Exceptionally, however, two or more hinges may be formed simultaneously. In a symmetrically loaded symmetric structure, for instance, the formation of a new hinge off the axis of symmetry will be accompanied by that of a symmetrically located hinge. In this particular case, the condition for the maintenance of one of these hinges will automatically ensure the maintenance of the other hinge, that is, the two new relations between attacking moments are not independent of each other. In general, when two or more hinges form simultaneously, the interdependence of the resulting relations between attacking moments must be investigated, and supernumerary relations must be discarded.

It will now be demonstrated that Step I can always be continued until Step II can be initiated. Assume first that no mechanism relation has as yet been established between the original attacking moments. A change of one attacking moment then affects only the full plastic moment of a single span. By choosing the appropriate sign, this change can therefore be made to save structural weight. For example, operation with line 9 in Table I reduces only M_A, and operation with line 13 reduces only M_B; thus, the structural weight can be reduced by operating with either line 9 or line 13.

Even after mechanism relations have been established between some attacking moments, such beneficial moves can still be made by operating with unrelated attacking moments.

Consider now a later stage in the computation, when there are no unrelated attacking moments left. In the example of Part I, this stage is reached with Eq. 16, which follows from the mechanism of Fig. 2*b*, when m_7 is set equal to m_B to maintain the hinge at section 7.

Inspection of Fig. 2*b* reveals that this equation can be written in the form

$$m_A\theta_A + m_B\theta_B = 0, \tag{18}$$

where θ_A and θ_B are the sums of the absolute values of the hinge rotations in spans A and B, respectively. As positive values of m_A and m_B correspond to a reduction of the full plastic moments M_A and M_B, the decrease in weight is

$$\Delta W = m_A l_A + m_B l_B, \tag{19}$$

where l_A and l_B are the lengths of the two spans. Of course, m_A and m_B cannot be given arbitrary values in (19) but must satisfy (18). If (19) yields a non-vanishing ΔW for some admissible choice of m_A and m_B, the signs of these attacking moments can be adjusted to furnish a decrease in weight. (This is done in line 16 of Table I.) If, on the other hand, ΔW should vanish, that is, if

$$m_A l_A + m_B l_B = 0 \tag{20}$$

for all admissible choices of m_A and m_B, comparison of (18) and (20) would indicate that the mechanism of Fig. 2*b* is a Foulkes mechanism, so that a minimum weight design would have been obtained even though this mechanism has only a single degree of freedom.

Returning now to the general discussion, assume that k mechanism relations have been established between the attacking moments and that there are no unrelated attacking moments left. This means, of course, that each of the N attacking moments occurs in at least one of the k mechanism relations.

For $k = N$, these independent linear relations admit only the trivial solution $m_A = m_B \cdots = m_N = 0$. In other words, the structural weight cannot be further reduced by a suitable choice of attacking moments. This is the signal for the initiation of Step II.

Exceptionally, more than one additional hinge may form in the last phase of Step I, and this may raise the number of mechanism relations for the attacking moments from a value below N to one exceeding N. Since, however, only N of these relations will be independent, the remarks made for $k = N$ remain valid.

For $k < N$, on the other hand, the mechanism relations admit non-vanishing attacking moments, and it must be shown that either these can be chosen to furnish a smaller structural weight or Step II can be started. Consider the expression for the decrease in weight ΔW brought about by a set of values of the attacking moments:

$$m_A l_A + m_B l_B + \cdots + m_N l_N = \Delta W. \tag{21}$$

The system of linear equations obtained by joining the nonhomogeneous Eq. 21 to $k < N$ independent mechanism relations will in general admit solutions for an arbitrarily chosen value of ΔW. In other words, it

will in general be possible to reduce the structural weight. (In the actual computation, the mechanism relations are not retained to form a system of k equations, but each new mechanism relation is immediately used to reduce the number of independent attacking moments. It is readily seen that this does not invalidate the conclusion just drawn: it will in general be possible to reduce the structural weight by a suitable choice of the independent attacking moments.)

It may happen, however, that (21) with a non-zero right-hand side contradicts the $k < N$ mechanism relations between the attacking moments. This means that the mechanism relations imply

$$m_A l_A + m_B l_B + \cdots + m_N l_N = 0. \tag{22}$$

The attacking moments can then no longer be chosen to reduce the structural weight. Because, however, the mechanism relations can be combined to yield (22), the corresponding mechanisms can be combined to yield a true or false Foulkes mechanism. Step II can therefore be initiated, even though the number of hinges obtained at this stage is less than that normally expected in a Foulkes mechanism.

Step II and Start of Subsequent Step I

Step II is initiated when the structural weight can no longer be reduced by the technique of Step I. As a rule, this situation arises when $k = N$ independent homogeneous linear relations have been established between the N attacking moments. Exceptionally, however, Step I may end for $k < N$, or more than one additional hinge may form in the last phase of Step I so that the number of mechanism relations for a hypothetical next phase would exceed N, though only N of these are independent.

In the regular case $k = N$, the N relations between the attacking moments correspond to N independent mechanisms, and these may be combined to yield a true or false Foulkes mechanism. In the first instance, a minimum weight design has been obtained at the end of the preceding Step I. In the second case, a new Step I is started by dropping one hinge of the wrong sign and thereby reducing to $N - 1$ the number of mechanism relations between attacking moments. Before it can be taken for granted, however, that the preceding discussion of Step I applies to this new Step I, the following difficulty must be resolved. At the hinge that is to be dropped, the absolute value of the bending moment equals the full plastic moment, but the absolute value of the residual moment is no longer set equal to the attacking moment, because it is not proposed to maintain the hinge. After the superposition of the residual moments, the absolute value of the bending moment at the dropped hinge might therefore exceed the common absolute value of the bending moment at the maintained hinges in the

same span. Only when it is shown that this cannot happen, is the feasibility of the new Step I established.

Consider a typical span, say K, with hinges at four sections and bending moments, hinge rotations, and attacking moments as shown in the following table:

bending moments:	M_K	M_K	$-M_K$	$-M_K$
hinge rotations:	(θ_1)	$(-\theta_2)$	$-(\theta_3)$	$-(-\theta_4)$
attacking moments:	$-m_K$	$-m_K$	m_K	m_K

The bending moments in the first line are those obtained at the end of the preceding Step I. The values θ_1 through θ_4 in the second line are positive; with the indicated signs they represent the hinge rotations of an intended Foulkes mechanism, the contents of the parentheses having been treated as positive quantities in applying Foulkes' criterion. Thus,

$$\theta_1 - \theta_2 + \theta_3 - \theta_4 = l_K. \tag{23}$$

The third line gives the attacking moments if the hinge that is to be dropped is in some other span so that all four hinges have to be maintained in the new Step I. On account of (23), the contribution of span K to the new mechanism relation is therefore given by

$$-m_K(\theta_1 - \theta_2 + \theta_3 - \theta_4) = -m_K l_K. \tag{24}$$

Now all possible combinations of positive or negative hinges with true or false Foulkes rotations are covered by the four columns of the table. The contribution (24) to the new mechanism relation is therefore typical for any span in which all previous hinges are maintained.

Consider next that the span K contains the hinge that is to be dropped, and assume for simplicity that this is the only hinge of wrong sign in its span. The table used above must then be enlarged by a fifth column referring to the hinge that is to be dropped. If this hinge is assumed to be positive, the entries in the fifth column would be the bending moment M_K, the false Foulkes rotation $(-\theta_5)$, and the residual moment m_5. At the same time, the term $-\theta_5$ would have to be added to the left-hand side of (23), or what amounts to the same, the right-hand side of this equation would have to be changed to $l_K + \theta_5$. The contribution of span K to the new mechanism relation then amounts to $-m_K(\theta_1 - \theta_2 + \theta_3 - \theta_4) - m_5\theta_5$, and the modified Eq. 23 shows this to equal

$$-(m_K + m_5)\theta_5 - m_K l_K. \tag{25}$$

Addition of the contributions from the other spans and use of (21) then yields the new mechanism relation in the form:

$$(m_K + m_5)\theta_5 + \Delta W = 0. \tag{26}$$

As an actual decrease in weight is desired ($\Delta W > 0$) and θ_s is positive by definition, (26) shows that

$$-m_K > m_s. \tag{27}$$

It follows that $M_K - m_K$, the new full plastic moment of the span K, exceeds $M_K + m_s$, the new bending moment at the dropped hinge. The dropping of a negative hinge of the wrong sign can be discussed in a similar manner; again, the new full plastic moment will exceed the absolute value of the bending moment at the dropped hinge.

It is readily seen, that the case $k < N$ does not create any additional difficulties. Similarly, the simultaneous formation of two or more hinges does not raise any new problems, provided that interdependent hinges are dropped simultaneously.

PART III

(a) *Lemmas*

1. *If a single point load of given magnitude acts on a beam, then the most unfavorable position of the load is at the center of the beam.*

If there is no hinge within the length of the beam, then the beam load does zero work on the Foulkes mechanism. If there is a hinge within the length of the beam, then this must occur under the load. Suppose the Foulkes mechanism involves hinge rotations in the beam as sketched in Fig. 3; the load acts at a distance x from the left-hand

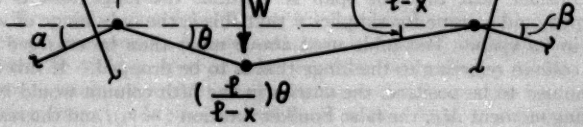

FIG. 3.

end of the beam, span l. As drawn in the figure, all hinge rotations occur within the length of the beam, and the total hinge rotation in the "span" of which the beam is a member is

$$\left[\alpha + \beta + \gamma + \left(\frac{2l}{l-x}\right)\theta\right], \tag{28}$$

where the term γ represents the sum of the hinge rotations in the portions of the structure, other than the beam under consideration, which

are being designed to the same full plastic moment. Thus, for minimum weight, expression (28) is proportional to the total length of the span under consideration, from which it will be seen that the hinge rotation θ must be proportional to $(l - x)$.

Now the work done by the load W is $Wx\theta$, and this quantity is thus proportional to $x(l - x)$. The work done by the load W is thus a maximum for $x = l/2$, that is, the load should act at midspan in order that the maximum work should be done on the collapse mechanism.

This discussion is essentially unaltered if one or both of the hinges at the ends of the beam occur on the far sides of the columns, that is, possibly in other spans, in which case their contributions to the total rotation in the span under consideration would be zero. However either α or β or both then become functions of x and θ, in order that the sums of hinge rotations in the other spans should satisfy Foulkes' criterion, and θ remains proportional to $(l - x)$.

2. *A beam load cannot do negative work on a collapse mechanism.*

Suppose negative work to be done; this implies a mechanism of the type sketched in Fig. 4, where W is essentially positive. From the

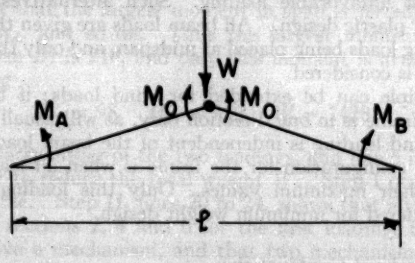

FIG. 4.

figure,

$$\frac{Wl}{2} = (M_A + M_B) - 2M_0. \tag{29}$$

But $|M_A| \leq M_0$, $|M_B| \leq M_0$; hence $W \leq 0$ from Eq. 29, which contradicts the assumption $W > 0$.

3. Suppose the loads on each beam of a multi-storey multi-bay frame can have values between prescribed positive (vertically downwards) limits. *The most unfavorable combination of beam loads occurs when all beam loads have their maximum positive values.* (This lemma is not intended to apply to *varying* loads, for which a shakedown design may be required.)

Suppose the minimum weight design is achieved with one load $W^* < W_{max}$. If there is no beam mechanism in the span on which W^* acts, the load does zero work on the minimum weight mechanism, and can have any value. If there is a beam mechanism, then W^* does positive work on this mechanism, by lemma 2. Hence the most unfavorable design will occur for $W^* = W_{max}$.

These lemmas have important practical consequences. First, a set of loads on a beam can be replaced by an equivalent load at midspan; the consequences of this approximation have been explored for the case of analysis (2,3,5). Secondly, for the general case of a frame subjected to a set of independently varying beam loads, all positive downwards, it is necessary to consider only the design for which all loads have their maximum values.

As a simple but important example of this principle, consider a continuous beam on several supports, each span of which can be subjected to independent vertical downward loads. The handbooks for elastic design give various load combinations which must be analyzed to derive maximum bending moments at different sections of the beam; alternatively, influence lines may be drawn and the loads positioned to give the most unfavorable loading. Such alternatives need not be considered for plastic design. All beam loads are given their maximum values, moving loads being placed at midspan, and only this one loading configuration is considered.

The principle can be extended for wind loads; if the horizontal loading on a frame is in one direction only, as will usually be the case, and if this wind loading is independent of the beam loading, then the worst loading configuration for the frame is when all loads, beam and wind, have their maximum values. Only this loading combination need be considered for minimum weight design.

(*b*) *Alternative Load Combinations*

The conclusion above holds only if the various loads acting on a frame are *independent* of each other. An important case of interdependent loads occurs in practice with variable load factor design. Supposing the loads V and H in Fig. 1 to represent the working values of dead plus superimposed and of wind loading respectively, it is common to design the frame under load V alone to a load factor λ_1, and under the loads V and H together to a load factor λ_2. The reduction in load factor from λ_1 to λ_2 expresses effectively the measure of unlikeliness of both loads acting together. Thus, working to $\lambda_1 = 1.75$, $\lambda_2 = 1.4$, the designer must investigate the collapse loading configurations $(1.75V, 0)$ and $(1.4V, 1.4H)$. In these two loading systems, the loads V and H are not independent.

Taking again the simple example of Fig. 1, suppose the frame is to be designed for working loads $V = 120$, $H = 60$, using load factors of 1.75 and 1.40. Thus the load combinations to be considered are $(V, H) \equiv (210, 0)$ and $(168, 84)$. The second load combination is the one already considered, Table I, for which the minimum weight design was a uniform frame having full plastic moment 98. It is easy to see that this frame will not carry the other loading combination, since, to carry the beam load $V = 210$, a minimum beam section $M_B = 105$ is required. Similarly, the minimum weight design for the first loading combination is $M_A = 0$, $M_B = 210$, supposing that a zero-section column is permitted; but this frame cannot resist the side load of the second load combination. Thus neither of the minimum weight designs, considering the two loading conditions separately, can be the design for the frame.

Table II shows the working for the example, which proceeds exactly as before except that both loadings are considered simultaneously. In line 5, Step "0," *two* equilibrium solutions are entered, one for each loading; in principle, any such solutions can be chosen, but line 5 corresponds to the minimum weight designs for each loading. The table of permissible changes, lines 6 to 9, is set up separately for each loading, except that the largest moment in span *A for either loading* is attacked by m_A, and so on. Thus the largest moment at sections 5, 6 and 7 (span B) is 210, and only this moment is attacked by m_B. A frame having

$$M_A = 98, \quad M_B = 210, \quad W = 1428 \tag{30}$$

will certainly carry either of the two loadings, and line 5 is modified as before in order to reduce the total weight W. Step I occupies lines 6 to 19 in the table. Step II, lines 20 to 24, shows that although hinges are formed at sections 2, 4 and 6 for the first loading, these are not sufficient to give a mechanism, and that two mechanisms are formed for the second loading. These mechanisms are the same as in lines 29 and 30 of Table I; as before, suppression of the hinge at section 2 enables another Step I to be made, lines 25 to 27 in Table II. In a further Step II, it is found that one mechanism is formed for each loading, lines 29 and 30. Taking multiples α and β of these, as before, line 31 shows that

$$\begin{aligned} 2\alpha + 4\beta &= 6 \\ 2\alpha + 2\beta &= 4 \end{aligned} \tag{31}$$

where, for example, $2\alpha + 4\beta$ is the total hinge rotation in span A. The solution $\alpha = \beta = 1$ gives the set of rotations, line 32, for which all hinges are of the correct sign, indicating that the minimum weight has been reached.

TABLE II.

STEP	LINE	PLASTIC MOMENT SECTION	M_A 1	2	3	4	M_B 5	6	7	M_A 1	2	3	4	M_B 5	6	7	
DATA	1		-1	1	-1	1											3H
	2						-1	2	-1								2V
	3			1			-1										0
	4				1				-1								0
0	5		0	0	0	0	0	210•	0	-98•	-42	-98•	98•	-42	98	-98	W=1428
I	6	6x35=210 m_A								1	3	1	-1	3	2	1	
	7	4x49=196 m_B		-2		2	-2	-1									
	8	x_1		-1	1	2	-1		1								
	9	x_2	1			1											
	10	m_A=35								35	105	35	-35	105	70	35	
	11		0	0	0	0	0	210	0	-63	63•	-63	63	63	168	-63	W=1218
I	12	4x31.5=126 m_B		-2		2	-2	-1									
	13	x_1		-1	1	2	-1		1								
	14	x_2	1			1											
	15	m_B=31.5		-63		63	-63	-31.5									
	16		0	-63•	0	63•	-63	178.5	0	-63	63	-63	63	63	168	-63	W=1092
	17	4x10.5=42 m_B	?		-2			-1	-2								
	18	m_B=10.5	21		-21			-10.5	-21								
	19		21	-63	-21	63	-63	168	-21	-63	63	-63	63	63	168•	-63	W=1050
II	20			-		+		+		-	+	-	+		+		
	21	α 1								-1	1	-1	1				
	22	β 2-3-4									-1	-1			2		
	23									-(α)	(α-β)	-(α+β)	(α)		(2β)		
	24	α=1.5; β=2								-(1.5)	(-0.5)	-(3.5)	(1.5)		(4)		
I	25	2x21=42 m_A	-1	1	3	1	1	2	3	1	3	1	-1	3	2	1	
	26	m_A=-21	21	-21	-63	-21	-21	-42	-63	-21	-63	-21	21	-63	-42	-21	
	27		42	-84	-84•	42	-84	126	-84	-84	0	-84	84	0	126	-84	W=1008
II	28			-	-			+		-		-	+		+		
	29	α 2-3-4		-1	-1			2									
	30	β 1+2-3-4								-1		-2	1		2		
	31			-(α)	-(α)			(2α)		-(β)		-(2β)	(β)		(2β)		
	32	α=1; β=1		-(1)	-(1)			(2)		-(1)		-(2)	(1)		(2)		

It is not necessary, in general, for a collapse mechanism to be found for each alternative loading. One or more of the loadings may be so "light" as to have no influence in determining the minimum weight. Even if the loadings are comparable in magnitude, the minimum weight design may still correspond to only one loading. In either of these cases, the method illustrated in Table II will generate the minimum weight design for one loading, while at the same time generating a permissible equilibrium solution for the other loading (or loadings).

It should be noted that if two loadings are considered as acting on a frame, the number of storage units for an electronic calculating machine is approximately doubled, and trebled for three loadings, and so on. Thus the maximum size of frame which can be handled by a given machine will be roughly halved if two loadings must be considered.

(*c*) *Limitation of Sections*

Table III(a) shows the solution of the same simple frame (Fig. 1) under the loading $(V, H) = (168, 84)$, under the assumption that the column sections are unequal, the total weight of the frame being given by

$$W = 3M_A + 4M_B + 3M_C. \tag{32}$$

Starting with line 17 of Table I as Step 0, the solution in line 4 of Table III

$$M_A = 10.5, \quad M_B = 115.5, \quad M_C = 115.5, \quad W = 840 \tag{33}$$

is the minimum weight, as can be seen from the compatible hinge rotations in line 10.

The frame is of course statically determinate at collapse, and the thrust in the column 1–2 is 52.5. Thus for the design of this column, this thrust acts on a section whose full plastic moment is 10.5. It may well be found that the section is inadequate when the stability of the column is considered; this may be checked quickly from charts given in the literature (2,10). A good estimate may also be made of the minimum section required for stability purposes for the column 1–2; suppose this estimate gives a minimum of 40 as the full plastic moment required for a thrust of 52.5. If this column section is used, there is the possibility of reducing the weight of the structure by redistributing material between M_A, M_B, and M_C.

Table III(*b*) uses as Step 0 the minimum weight solution, line 4 of Table III(*a*); the weight of the frame in Step 0 is 928.5, since it is assumed that $M_A = 40$. Thus hinges no longer occur at sections 1 and 2. Permissible changes are shown in line 12; note that no material loss is incurred in span A until the total moment at section 1 or 2 exceeds 40. Line 14 gives a new design of frame, with

$$M_A = 40, \quad M_B = 109.6, \quad M_C = 109.6, \quad W = 887.2 \tag{34}$$

and Step I is complete since no further weight-saving change may be made. There are two independent mechanisms of collapse, lines 16 and 17, and these can be combined to satisfy Foulkes' criterion in two spans only. Line 19 shows the criterion satisfied for spans B and C, and that the hinge rotation in span A at section 1 is of the right sign. This illustrates the application of Foulkes' criterion when there are restrictions on the values of the full plastic moments of one or more

TABLE III.

STEP	LINE	PLASTIC MOMENT SECTION		M_A 1	M_A 2	M_C 3	M_C 4	M_B 5	M_B 6	M_B 7	w=960
0	1			-98*	-42	-98*	98*	-42	98*	-98*	
I	2	8×17.5=140	m_B	-5	-3	1	-1	-3	-1	1	
	3	m_B=-17.5		87.5	52.5	-17.5	17.5	52.5	17.5	-17.5	
	4			-10.5	10.5*	-115.5	115.5	10.5	115.5	-115.5	w=840
II	5			-	+	-	+		+	-	
	6	α	1	-1	1	-1	1				
	7	β	2-3		-1				2	-1	
	8	γ	4			1				-1	
	9			-(α)	(α-β)	-(α-γ)	(α)		(2β)	-(β+γ)	
	10	α=2;β=1;γ=1		-(2)	(1)	-(1)	(2)		(2)	-(2)	

(a)

STEP	LINE			M_A 1	M_A 2	M_C 3	M_C 4	M_B 5	M_B 6	M_B 7	w=928.5
0	11			-10.5	10.5	-115.5*	115.5*	10.5	115.5*	-115.5*	
I	12	7×5.9=41.3	m_B	-5	-3	1	-1	-3	-1	1	
	13	m_B=5.9		-23.5	-17.7	5.9	-5.9	-17.7	-5.9	5.9	
	14			-10*	-7.2	-109.6	109.6	-7.2	109.6	-109.6	w=867.2
II	15			-		-	+		+	-	
	16	α	1+2+3	-1		-1	1		2	-1	
	17	β	4			1				-1	
	18			-(α)		-(α-β)	(α)		(2α)	-(α+β)	
	19	α=1.4;β=-0.2		-(1.4)		-(1.6)	(1.4)		(2.8)	-(1.2)	

(b)

spans; the criterion must be satisfied for the "freely" designed spans, but not necessarily satisfied (except by accident) for those spans with an imposed minimum section.

The thrust in column 1–2 corresponding to the solution in line 14 is 58.4, compared with 52.5 previously. This small change in thrust is characteristic of frames of the type discussed here, and implies that column design can be a quickly converging process.

It may be noted that a restriction on the size of a section is always of the form $M_K \geq M_K^0$, where M_K^0 is the minimum value of the full plastic moment that is allowed for span K. If at the end of Step I it is found that $M_K = M_K^0$, that is, that the limitation is *effective*, then the equation $M_K = M_K^0$ can be treated as a mechanism equation for the purpose of satisfying Foulkes' criterion.

From Table III, lines 16 and 17 give

$$\begin{aligned} \alpha M_A + 2\alpha M_C + 3\alpha M_B &= 588\alpha \\ -\beta M_C + \beta M_B &= 0 \end{aligned} \qquad (35)$$

where the number 588 is the work done by the loads on the mechanism

of line 16. If to these two equations is added

$$\gamma M_A = 40\gamma \tag{36}$$

representing the fact that M_A has its minimum permissible value, then the three equations may be superimposed in the proportions $\alpha = 7/5$, $\beta = -1/5$, $\gamma = 8/5$ to give hinge rotations in the spans A, B, and C of 3, 4, and 3 respectively, that is, to give hinge rotations satisfying Foulkes' criterion.

(*d*) *Interdependent Hinges*

Step 0 in Table IV refers to the same frame, Fig. 1, with $(V, H) = (100, 200)$, and with the two column sections equal as before. Step 0

TABLE IV.

STEP	LINE	PLASTIC MOMENT SECTION			M_A 1	2	3	4	M_B 5	6	7	
0	1				-300∘	-300∘	-300∘	300∘	-300∘	-200	-300∘	W=3000
	2				-	-	-	+	-		-	
	3	α	1		-1	1	-1	1				
II	4	β	3			1			-1			
	5	γ	4				1				-1	
	6				-(α)	-(-α-β)	-(α-γ)	(α)	-(β)		-(γ)	
	7	α=5; β=4-γ			-(5)	-(γ-9)	-(5-γ)	(5)	-(4-γ)		-(γ)	
	8	10×150=1500		m_A	1	3	1	-1	3	2	1	
I	9	m_A=150			150	450	150	-150	450	300	150	
	10				-150	150∘	-150	150	150∘	100	-150	W=1500
	11				-	+	-	+	+		-	
II	12				-(α)	-(α+β)	-(α-γ)	(α)	-(β)		-(γ)	
	13	α=2.5; β=γ-4			-(2.5)	(γ-1.5)	(2.5-γ)	(2.5)	(4-γ)		(γ)	

in fact represents a mechanism, and no Step I can be made. Step II shows three alternative mechanisms, and line 7 represents the Foulkes mechanism for which the hinge rotations in span A total 6, and those in span B, 4. Notice that γ cannot be eliminated from these hinge rotations. There are only two equations for three unknowns; these equations yield

$$\begin{aligned} \alpha &= 5 \\ \beta + \gamma &= 4. \end{aligned} \tag{37}$$

However, it can be seen that line 7 cannot possibly give a true Foulkes mechanism; there is no value of γ to give hinge rotations of the correct sign at sections 2 and 3 simultaneously.

The rule is to "cancel" *two* hinges of the wrong sign which are interdependent in this way, that is, those at sections 2 and 3, or at sections

2 and 5, for example, but *not* at sections 3 and 5, since for these last $\gamma < 4$ would be satisfactory at these two hinges alone.

Line 8 gives the permissible change consequent on releasing the restrictions on the hinges at sections 2 and 5, and leads to the correct solution, line 10. Notice that Foulkes' mechanism, line 13, still contains interdependent hinges, but that a value of γ can be found (for example, $\gamma = 2$), to give hinge rotations of the right signs.

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ON THE NUMERICAL SOLUTION OF n-DIMENSIONAL BOUNDARY VALUE PROBLEMS ASSOCIATED WITH POISSON'S EQUATION

BY

DONALD GREENSPAN [1]

1. INTRODUCTION

Recently there has been a revival of interest in n-dimensional elliptic differential equations (1, 2, 3, 4, 5).[2] However, the analytical determination of solutions of associated boundary value problems, when they do exist (6, 7), usually offers what are at present insurmountable problems. Hence, numerical methods have become quite popular for such problems (8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18) and the present intent is to further explore this approach.

Let G be a closed, bounded, simply connected n-dimensional region of n-dimensional Euclidean space whose interior is denoted by R and whose bounding hypersurface is denoted by S. Let $g(x_1, x_2, \cdots, x_n)$ be defined and continuous on S and let $f(x_1, x_2, \cdots, x_n)$ be defined and of class C^2 on G. The problem then is as follows: Produce a function $u(x_1, x_2, \cdots, x_n)$ such that

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f(x_1, x_2, \cdots, x_n) \tag{1}$$

on G, and

$$u(x_1, x_2, \cdots, x_n) \equiv g(x_1, x_2, \cdots, x_n) \tag{2}$$

on S.

Under quite general conditions, the problem possesses a unique solution (6, 7) and only such cases will be considered here.

Finally, it is worth noting that for the special case $f(x_1, x_2, \cdots, x_n) \equiv 0$ the discussion yields the numerical solution of the n-dimensional Dirichlet problem.

2. NUMERICAL METHOD

Let $h_1, h_2, \cdots, h_n$ be fixed positive constants and let $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ be an arbitrary, but fixed, point of G. Denote by G_h the set of all points of the form $(\bar{x}_1 + m_1 h_1, \bar{x}_2 + m_2 h_2, \cdots, \bar{x}_n + m_n h_n)$ contained in G, where $m_1, m_2, \cdots, m_n$ are integers. Two points $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$ of G_h are called adjacent if and only if the straight line segment joining the two points is contained entirely in G, and

$$\sum_{i=1}^{n} (x_i - y_i)^2 = h_j^2,$$

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

where $x_i = y_i$ for all i *except* $i = j$, and $j = 1, 2, 3, \cdots, n-1$, or n. The interior of G_h, denoted by R_h, is the set of all points of G_h which have $2n$ adjacent points which also belong to G_h. The boundary of G_h, denoted by S_h, and called the lattice boundary, is defined by $S_h = G_h - R_h$.

The numerical method, then, proceeds as follows. Suppose G_h consists of t points. Number these points in a one to one fashion with the integers $1, 2, 3, \cdots, t$. Denote the coordinates of the point numbered k by $(x_1, x_2, \cdots, x_n)_k$ and the unknown function u at $(x_1, x_2, \cdots, x_n)_k$ by $u(x_1, x_2, \cdots, x_n)_k = u_k$, for $k = 1, 2, 3, \cdots, t$.

First suppose that $(x_1, x_2, \cdots, x_n)_i$ represents an arbitrary point of S_h. Approximate u_i by $u_i = g(x_1', x_2', \cdots, x_n')$, where $(x_1', x_2', \cdots, x_n')$ is the nearest point of S to $(x_1, x_2, \cdots, x_n)_i$. If $(x_1', x_2', \cdots, x_n')$ is not unique, choose any one of the set of nearest points and use it. The problem of finding numerical approximations to $u(x_1, x_2, \cdots, x_n)$ on the lattice boundary is, though crudely done, adequate for present purposes.

We then require that at each point $(x_1, x_2, \cdots, x_n)$ of R_h, the function u satisfy:

$$
\begin{aligned}
&-2u(x_1, x_2, \cdots, x_n) \\
&+ \frac{1}{\sum_{i=1}^{n} B_j} \sum_{j=1}^{n} \{B_j[u(x_1, x_2, \cdots, x_{j-1}, x_j + h_j, x_{j+1}, \cdots, x_n) \\
&+ u(x_1, x_2, \cdots, x_{j-1}, x_j - h_j, x_{j+1}, \cdots, x_n)]\} - \frac{h_1^2 h_2^2 \cdots h_n^2}{\sum_{i=1}^{n} B_j} \\
&\times f(x_1, x_2, \cdots, x_n) = 0, \qquad (3)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= h_2^2 h_3^2 h_4^2 \cdots h_n^2 \\
B_n &= h_1^2 h_2^2 \cdots h_{n-2}^2 h_{n-1}^2 \qquad (4) \\
B_j &= h_1^2 h_2^2 \cdots h_{j-2}^2 h_{j-1}^2 h_{j+1}^2 h_{j+2}^2 \cdots h_n^2, \; j = 2, 3, \cdots, n-1.
\end{aligned}
$$

Application of difference Eq. 3 to each point of R_h results in a system of linear, algebraic equations which, when solved, yields the remaining numerical approximations. In the remainder of the discussion, it will be shown that this linear system has a unique solution and that, under rather general conditions, the numerical solution converges to the analytic solution. As for the derivation of Eq. 3, if one constructs the equations for the special cases $n = 2$ and $n = 3$, as described in (8, pp. 153–155), then the general form given above readily suggests itself.

3. THEOREMS ON THE NUMERICAL METHOD

In this section, let $u(x_1, x_2, \cdots, x_n)$ represent the analytic solution of the boundary value problem described in Section 1 and let $U(x_1, x_2, \cdots, x_n)$ represent the numerical solution. This latter assumption means that if $(x_1, x_2, \cdots, x_n)$ is a point of R_h, then

$$-2U(x_1, x_2, \cdots, x_n) + \frac{1}{\sum_{j=1}^{n} B_j} \sum_{j=1}^{n} \{B_j[U(x_1, x_2, \cdots, x_j + h_j, x_{j+1}, \cdots, x_n) + U(x_1, x_2, \cdots, x_{j-1}, x_j - h_j, x_{j+1}, \cdots, x_n)]\} - \frac{h_1^2 h_2^2 \cdots h_n^2}{\sum_{j=1}^{n} B_j} \times f(x_1, x_2, \cdots, x_n) = 0. \quad (5)$$

Theorem 1

The solution of the system of linear equations which results by application of the numerical method of Section 2 is unique.

Proof. It is sufficient to show that the determinant of the system of linear equations is not zero and this is done by demonstrating that the only solution of the homogeneous system which results by considering $g(x_1, x_2, \cdots, x_n) \equiv 0$ on S and $f(x_1, x_2, \cdots, x_n) \equiv 0$ on G is the zero solution. It follows immediately from the numerical method that the choice of $f(x_1, x_2, \cdots, x_n)$ does not affect the determinant of the resulting linear system so that, for present purposes, one may allow it to be identically zero on G.

Suppose then that there exists a non-trivial solution for the homogeneous system. For some point of R_h, $U \neq 0$. Suppose, without loss of generality, $U > 0$. Let the largest value M occur at $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$. Note that this is not the same $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ as that described in Section 2, since this is a fixed point. Then:

$$2U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = \frac{1}{\sum_{j=1}^{n} B_j} \sum_{j=1}^{n} \{B_j[U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_j + h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n) + U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-2}, \bar{x}_{j-1}, \bar{x}_j - h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n)]\} \quad (6)$$

and

$$M = U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \geq U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_j \pm h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n); \; j = 1, 2, \cdots, n. \quad (7)$$

We need then consider only two cases.

Case 1. Suppose all the $U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_j \pm h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n)$; $j = 1, 2, \cdots, n$, are equal. Then from Eqs. 6 and 7, it follows that

$$M = U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = U(\bar{x}_1, + h_1, \bar{x}_2, \cdots, \bar{x}_n).$$

Case 2. Suppose not all the $U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_j \pm h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n)$ are equal. Then there exists a maximum. Suppose, without loss of generality, $U(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n)$ is that maximum. A similar argument follows for any other choice. Hence

$$\begin{aligned}
&U(\bar{x}_1 - h_1, \bar{x}_2, \cdots, \bar{x}_n) = U(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n) - \alpha_1, \\
&U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_j + h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n) \\
&\qquad = U(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n) - \alpha_j; \quad j = 2, 3, \cdots, n, \qquad (8) \\
&U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1}, \bar{x}_j - h_j, \bar{x}_{j+1}, \cdots, \bar{x}_n) \\
&\qquad = U(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n) - \beta_j; \quad j = 2, 3, \cdots, n,
\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n, \beta_2, \beta_3, \cdots, \beta_n$ are non-negative, and at least one is positive. Substitution of Eqs. 8 into Eq. 6 yields:

$$2U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = 2U(\bar{x}_1 + h, \bar{x}_2, \bar{x}_3, \cdots, \bar{x}_n) - S; S > 0.$$

However, this implies that $U(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n) > U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, which contradicts Eq. 7. Hence, Case 2 is not possible.

From the discussion of Cases 1 and 2, it follows that:

$$U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = U(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n) = M.$$

Repeating the argument at $(\bar{x}_1 + h_1, \bar{x}_2, \cdots, \bar{x}_n)$, one may then continue, in a finite number of steps, to show that U at a boundary point is equal to M. But this is also a contradiction, since $g(x, y) \equiv 0$ on S implies that U is zero at each point of S_A. Hence the theorem is proved.

We proceed by generalizing certain methods and results first proved by Gerschgorin (17).

Let:

$$\begin{aligned}
L[v(x_1, x_2, \cdots, x_n)] \equiv \frac{\sum_{j=1}^{n} B_j}{h_1^2 h_2^2 \cdots h_n^2} \Bigg(&- 2v(x_1, x_2, \cdots, x_n) \\
&+ \frac{1}{\sum_{j=1}^{n} B_j} \sum_{i=1}^{n} \{B_j[v(x_1, x_2, \cdots, x_{j-1}, x_j + h_j, x_{j+1}, \cdots, x_n) \\
&\qquad + v(x_1, x_2, \cdots, x_{j-1}, x_j - h_j, x_{j+1}, \cdots, x_n)]\} \Bigg). \qquad (9)
\end{aligned}$$

Lemma 1. If $L[v] \leq 0$ on R_h, and $v \geq 0$ on S_h, then $v \geq 0$ on R_h. The proof follows easily as in (8, p. 156).

Lemma 2. If $-|L[v_1]| \geq L[v_2]$ on R_h and $|v_1| \leq v_2$ on S_h, then $|v_1| \leq v_2$ on R_h.
The proof again follows easily as in (8, p. 156).

Lemma 3. If $|L[v]| \leq A$ on R_h and if $|v| \leq B$ on S_h and if r is the radius of an n-dimensional sphere which contains G, then on R_h:

$$|v| \leq \frac{Ar^2}{4} + B.$$

Proof. Let

$$w(x_1, x_2, \cdots, x_n) = \left[\frac{Ar^2}{4}\left\{1 - \frac{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}{r^2}\right\} + B\right],$$

where $(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = r^2$ is the equation of an n-dimensional sphere which contains G.

Direct calculation yields: $L[w] = -\frac{nA}{2}$. Since $n \geq 2$, it follows that $L[w] \leq -A$. Also, $w \geq B$ on S_h. Now, since $|L[v]| \leq A$ on R_h, by assumption, and $L[w] \leq -A$, it follows that on R_h $|L[v]| \leq -L[w]$. Also, $|v| \leq B$ on S_h and $w \geq B$ on S_h, so that $w \geq |v|$ on S_h. Hence: $-|L[v]| \geq L[w]$ on R_h and $w \geq |v|$ on S_h. By Lemma 2, then, $w \geq |v|$ on R_h, or, $|v| \leq w \leq \frac{Ar^2}{4} + B$, on R_h, and the lemma is proved.

Theorem 2

If u is of class C^4 and represents the analytic solution of the boundary value problem described in Section 1, U denotes the solution of the linear algebraic system described in Section 2, r is the radius of an n-dimensional sphere which contains G, and

$$M_i = \max_{\sigma}\left|\frac{\partial^i u}{\partial x_j{}^i}\right|; \; i = 1, 4; \; j = 1, 2, 3, \cdots, n,$$

then if $(x_1, x_2, \cdots, x_n)$ is any point of R_h, it follows that

$$|U(x_1, x_2, \cdots, x_n) - u(x_1, x_2, \cdots, x_n)| \leq \frac{M_4 r^2}{48}(h_1^2 + h_2^2 + \cdots + h_n^2) + M_1(h_1 + h_2 + \cdots + h_n). \quad (10)$$

Proof. Let

$$Q = f(x_1, x_2, \cdots, x_n) - L[u]. \tag{11}$$

Substitution of finite Taylor expansions for $u(x_1, x_2, \cdots, x_{j-1}, x_j + h_j, x_{j+1}, \cdots, x_n)$ and $u(x_1, x_2, \cdots, x_{j-1}, x_j - h_j, x_{j+1}, \cdots, x_n)$; $j = 1, 2, 3, \cdots, n$; into $L(u)$ and use of Eq. 1 transforms Eq. 11 to:

$$\begin{aligned} Q = -\Bigg[& \frac{h_1^2}{4!}\frac{\partial^4 u(\xi_1, x_2, \cdots, x_n)}{\partial x_1^4} + \frac{h_1^2}{4!}\frac{\partial^4 u(\xi_1', x_2, \cdots, x_n)}{\partial x_1^4} \\ & + \frac{h_2^2}{4!}\frac{\partial^4 u(x_1\ \xi_2, x_3, \cdots, x_n)}{\partial x_2^4} + \frac{h_2^2}{4!}\frac{\partial^4 u(x_1, \xi_2', x_3, \cdots, x_n)}{\partial x_2^4} \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + \frac{h_n^2}{4!}\frac{\partial^4 u(x_1, \cdots, x_{n-1}, \xi_n)}{\partial x_n^4} + \frac{h_n^2}{4!}\frac{\partial^4 u(x_1, \cdots, x_{n-1}, \xi_n')}{\partial x_n^4}\Bigg]. \end{aligned} \tag{12}$$

Hence:

$$|Q| \leq \frac{M_4}{12}(h_1^2 + h_2^2 + \cdots + h_n^2). \tag{13}$$

Moreover, it follows from Eqs. 5, 9 and 13 that:

$$\begin{aligned} |L[U-u]| &= |L[U] - L[u]| = |f(x_1, \cdots, x_n) - L[u]| \\ &= |Q| \leq \frac{M_4}{12}(h_1^2 + h_2^2 + \cdots + h_n^2). \end{aligned} \tag{14}$$

Recall too, that for any point of S_h, U was chosen equal to $g(x_1', x_2', \cdots, x_n')$ at a nearest point $(x_1', x_2', \cdots, x_n')$ on the boundary S, and $g(x_1', x_2', \cdots, x_n') = u(x_1', x_2', \cdots, x_n')$ on S. Hence, for any point $(x_1, x_2, \cdots, x_n)$ of S_h:

$$\begin{aligned} & |U(x_1, \cdots, x_n) - u(x_1, \cdots, x_n)| \\ & = |g(x_1', x_2', \cdots, x_n') - u(x_1, x_2, \cdots, x_n)| \\ & = |u(x_1', x_2', \cdots, x_n') - u(x_1, x_2, \cdots, x_n)| \\ & \leq \left|h_1 \frac{\partial}{\partial x_1} u(\gamma_1, \gamma_2, \cdots, \gamma_n)\right| \\ & + \left|h_2 \frac{\partial}{\partial x_2} u(\gamma_1, \gamma_2, \cdots, \gamma_n)\right| \\ & + \left|h_3 \frac{\partial}{\partial x_3} u(\gamma_1, \gamma_2, \cdots, \gamma_n)\right| \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + \left|h_n \frac{\partial}{\partial x_n} u(\gamma_1, \gamma_2, \cdots, \gamma_n)\right| \\ & \leq M_1(h_1 + h_2 + \cdots + h_n). \end{aligned} \tag{15}$$

Applying Lemma 3 to inequalities 14 and 15 yields inequality 10, and the theorem is proved.

Theorem 3

Under the conditions of Theorem 2, the numerical solution U converges to the analytic solution u as $(h_1^2 + h_2^2 + \cdots + h_n^2)$ converges to zero.

The proof for points of R_h follows immediately from inequality 10 while that for points of S_h follows directly from the numerical procedure described for such points.

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Chemical Separations Process for Spent Reactor Fuel.—Research on a chemical separations process for spent reactor fuel was revealed at the second international atoms for peace conference in Geneva. A General Electric chemist from the Atomic Energy Commission's Hanford plant said the process will be of special importance to possible future plutonium recycle power reactors.

The chemist, Dr. A. S. Wilson, described laboratory experiments using tertiary amine solutions to extract plutonium at the General Electric operated plant in Washington state. He said the process is inexpensive, and the chemicals involved are especially good because they are not affected by radiation.

The process described by Wilson uses a nitrogen-containing compound dissolved in kerosene. Wilson said the tertiary amine extraction process was first examined at Hanford for the purpose of selectively separating neptunium from certain waste solutions. It was discovered that plutonium could also be extracted effectively with the same process.

Wilson said a big advantage of the tertiary amine solution is its ability to selectively extract plutonium for re-use from spent fuel elements dissolved in nitric acid leaving everything else in solution. Also, he said that those power reactor fuels from which the depleted uranium is to be discarded and the plutonium recovered are uniquely suited to the reprocessing with a tertiary amine solvent extraction system, because this solvent separates plutonium from uranium without the necessity for many repeated separation steps now needed in existing production separation processes. The tertiary amine solutions do not extract aluminum, iron, nickel, and chromium nitrates which makes them ideally suited for the separation of plutonium from solutions of plutonium-aluminum alloys as well as from the dissolution products of stainless steel clad fuel elements.

The recovery of plutonium from highly radioactive waste solutions is made feasible, Wilson said, by the radiation resistance of the extractant. The radiation resistance of the solvent also means that continual re-use of the solvent is possible without extensive recovery processing.

Liquid Nitrogen Generator.—A new ADL Liquid Nitrogen Generator System manufactured by Arthur D. Little, Inc., of Cambridge, Mass., provides a complete nitrogen liquefying facility. Operation requires only pushbutton start and stop, and automatic controls permit extended operation without attention. Major components include the ADL Nitrogen Column, modified Norelco® Gas Liquefier and 200-liter storage tank, automatically providing up to 95 liters daily of 99.5% pure liquid nitrogen. This system proves to be economical to industrial and laboratory users alike who have moderate demands for liquid nitrogen.

In operation, the blower of the nitrogen column supplies oil-free air at a slight positive pressure. Water vapor and carbon dioxide are frozen out in the heat exchanger. Air is rectified in a packed column, and the nitrogen gas is passed from the top of the column to the modified condensing head of the Norelco liquefier. The high-purity nitrogen is liquefied, and the noncondensible gases are continuously bled off. A portion of the liquid is used as a reflux to cool the column and the rest is drawn off as product to a storage tank or transfer dewar.

V.2
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INERTIAL NAVIGATION

BY

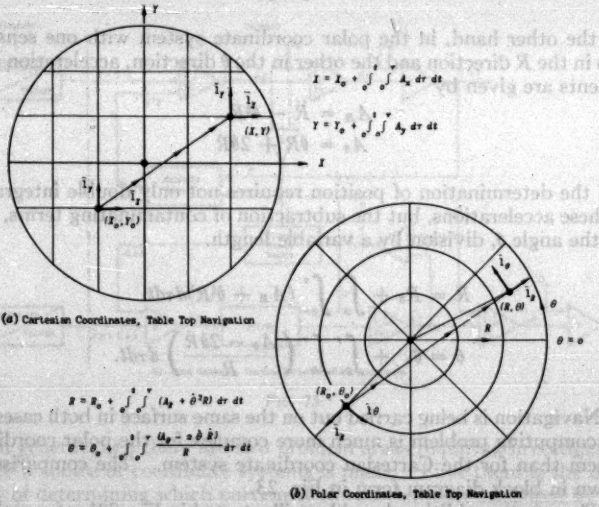
C. F. O'DONNELL [1]

Part II [2]

D. PROBLEMS OF INERTIAL GUIDANCE SYSTEMS

1. Coordinate Systems

At first glance the choice of a coordinate system, provided the origin and destination are known in terms of coordinates of that system, appears inconsequential. In practice the choice has a major bearing on the system mechanization, especially in the case of the position computer. Consider the two choices of coordinate systems for navigation on a circular table top shown in Fig. 22. Figure 22*a* shows a Cartesian



(*a*) Cartesian Coordinates, Table Top Navigation

(*b*) Polar Coordinates, Table Top Navigation

FIG. 22.

coordinate system, origin at the table center, with position measured in terms of x and y coordinates. Figure 22*b* shows a polar system, again

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[2] Part I appeared in this JOURNAL for October, 1958.

with the origin at the center of the table, and position measured in terms of R and θ, a distance and an angle. In many ways the choice of the polar coordinate system appears more reasonable for it matches the boundary conditions of the surface on which we are navigating more elegantly than do the Cartesian coordinates. However, in navigator mechanization if the accelerometer platform is aligned in the Cartesian system, one sensitive axis in the x direction, the other sensitive axis in the y direction, then

$$A_x = \ddot{X}$$
$$A_y = \ddot{Y}$$

and position is given simply by

$$X = X_0 + \int_0^t \int_0^\tau A_x d\tau dt$$
$$Y = Y_0 + \int_0^t \int_0^\tau A_y d\tau dt.$$

On the other hand, in the polar coordinate system with one sensitive axis in the R direction and the other in the θ direction, acceleration components are given by

$$A_R = \ddot{R} - \dot{\theta}^2 R$$
$$A_\theta = \theta R + 2\dot{\theta}\dot{R}$$

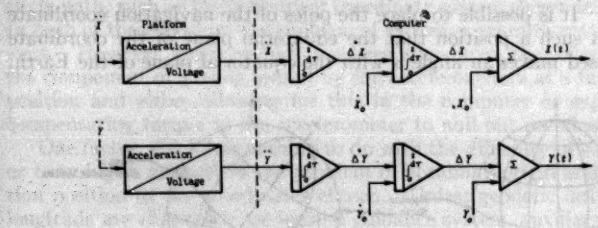
and the determination of position requires not only double integration of these accelerations, but the subtraction of contaminating terms, and, for the angle θ, division by a variable length.

$$R = R_0 + \int_0^t \int_0^\tau (A_R + \dot{\theta}^2 R) d\tau dt$$
$$\theta = \theta_0 + \int_0^t \int_0^\tau \left(\frac{A_\theta - 2\dot{\theta}\dot{R}}{R} \right) d\tau dt.$$

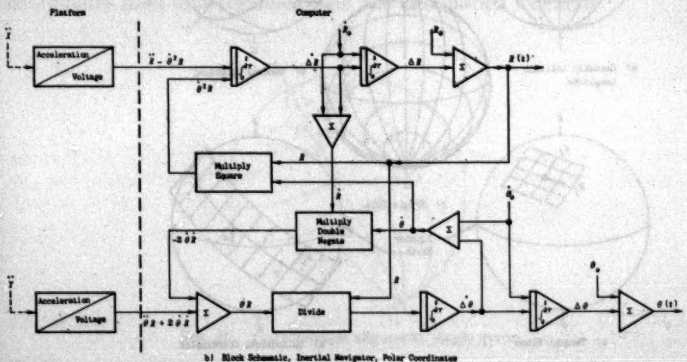
Navigation is being carried out on the same surface in both cases but the computing problem is much more complex for the polar coordinate system than for the Cartesian coordinate system. The comparison is shown in block diagram form in Fig. 23.

There is the additional problem, illustrated in Fig. 22*b*, of a variable azimuth rotation rate in the polar coordinate system. Here a flight passing close to the pole at the origin can require a swing in azimuth of almost 180° in a very short period of time. This not only taxes the capacity of the computer, but in case the gyro package is being torqued to rotate with the accelerometer platform, it increases the maximum

precession rate gyro and torquer must handle accurately. This problem of a variable azimuth torque is not present in the Cartesian coordinate system, therefore, in this particular example it can be seen that computer mechanization, torquing rate capability of the reference gyros, and accelerometer platform hardware are all strongly influenced by the choice of position coordinates.



a) Block Schematic, Inertial Navigator, Cartesian Coordinates.



b) Block Schematic, Inertial Navigator, Polar Coordinates

FIG. 23.

In general then there are two problem areas requiring investigation before a choice of coordinate systems is made. First there is the difficulty of determining which portion of the acceleration measured by the physical instruments is associated with a change in the position coordinate of interest, that is, x and y for Cartesian coordinates, R and θ for polar coordinates. Secondly there is the problem of maintaining the correct orientation of the accelerometer platform and/or the gyro reference package.

A number of possible coordinate systems are shown in Fig. 24.

The most widely known is that shown in Fig. 24*a*, a geodetic latitude-longitude system. Here the north and south poles are singular points. At the north pole itself longitude loses all meaning as a position coordinate and becomes only a direction in which to point. In this case the coordinate system poles coincide with the poles of the spin axis of the Earth.

Actually it is not necessary to have this coincidence, as shown in Fig. 24*b*. It is possible to place the poles of the navigation coordinate system in such a position that the equatorial plane of the coordinate system used makes an angle α with the equatorial plane of the Earth.

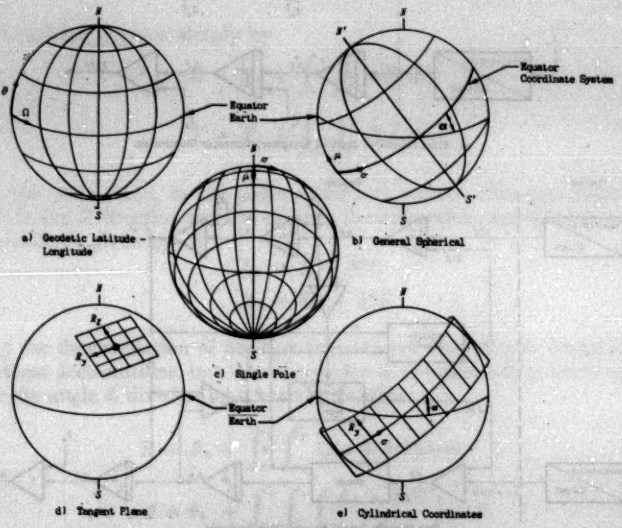


FIG. 24. Coordinate systems.

The purpose of this would be to avoid the problems of a rapid azimuth rotation of the accelerometer platform when approaching the Earth's north or south pole. The coordinate poles could be placed, for example, on the equator in an area in which the vehicle using the inertial navigator would not normally operate.

Another possibility would be the use of a single pole system shown in Fig. 24*c*. In this case only one of the Earth's actual poles becomes a singular point. It is perhaps easiest to visualize this cordinate system as being formed by a set of planes fanned out in two directions at right angles to each other about the south pole. This forms a true orthogonal

set of coordinates and allows flight at any point over the Earth other than the single pole area without difficulties in computation or generation of azimuth rates.

The coordinate systems so far considered have all been of the local level type. It is also possible to employ coordinate systems for which the platform will be level either only at a point, as in the case of the tangent plane system shown in Fig. 24*d* or along a particular great-circle path, such as the cylindrical coordinate system shown in Fig. 24*e*. In both of these systems the problem of excessive azimuth rate does not exist. However, a new problem has been introduced, that of computing the component of g being sensed by the accelerometers as a function of position and either allowing for this in the computer or supplying a compensating torque to the accelerometer to null out its effect.

One factor which has nothing to do with the difficulty of computing or torquing in azimuth is the problem of obtaining origin and destination position in the coordinates chosen. Unless geodetic latitude and longitude are chosen for the inertial guidance system, auxiliary calculations must be performed to convert from latitude-longitude to the coordinates used to determine origin and destination locations.

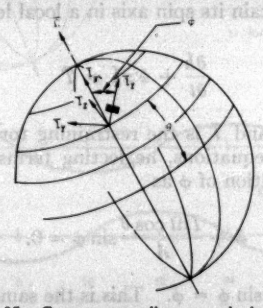


FIG. 25. Gyrocompass alignment, single gyro.

2. *Alignment*

Throughout this paper continual mention is made of the fact that the accelerometer platform must be correctly aligned. During system operation alignment is maintained through feedback loops existing between the platform and the computer. However, there remains the problem of establishing the initial alignment of the system. One possibility is the use of an optical system employing a theodolite, and an external reference for aligning the platform both in azimuth and level. An alternate possibility is to make use of the accelerometers and gravity to level the platform, with the optical system and an external reference being employed only for azimuth alignment. If it is required

that the system be completely self-contained, an azimuth reference can be obtained from the Earth rate vector Ω while gravity is used to establish the local vertical.

In the traditional gyrocompassing method, the platform would be held level using the gravity vector as a reference for the platform accelerometers and a gyroscope lined up so that its spin axis was approximately north-south (Fig. 25). Then as the gyro is restrained to move in a locally level plane and as, through the Earth's rotation, this plane is moving with respect to inertial space, the total angular rate of the gyro is as follows:

$$\bar{\omega} = (\Omega \cos\phi \cos\theta + s)\bar{I}_x \Omega \sin\phi \cos\theta \bar{I}_y + (\dot{\phi} + \Omega \sin\theta)\bar{I}_z$$

where s is the rotor spin rate, ϕ the misalignment angle, and θ the latitude. Its angular momentum is:

$$\bar{h} = I(\Omega \cos\phi \cos\theta + s)\bar{I}_x - A\Omega \sin\phi \cos\theta \bar{I}_y + A(\dot{\phi} + \Omega \sin\theta)\bar{I}_z$$

where I is the rotor axial moment of inertia and A is its transverse moment of inertia. Then allowing for the fact that the gyro is being torqued only to maintain its spin axis in a local level plane and that in general:

$$\frac{\partial \bar{h}}{\partial t} + \bar{\omega}' x \bar{h} = \bar{T}$$

where $\bar{\omega}' = \bar{\omega} - s\bar{I}_x$, and $\bar{T}$ is the restraining torque; and solving the resultant differential equations, neglecting terms in Ω^2, gives the expression for determination of ϕ as

$$\ddot{\phi} + \frac{Is\Omega \cos\theta}{A} \sin\phi = 0.$$

For small values of ϕ, $\sin\phi = \phi$. This is the same form of equation as that governing error propagation in inertial navigators. The undamped harmonic oscillations in this case have a period $\tau = 2\pi\sqrt{\dfrac{A}{Is\Omega \cos\theta}}$. If no disturbing torques are present the platform under the control of this gyro oscillates about the true north direction with a peak amplitude equal to the original misalignment angle. Either a fixed reference can be employed, the mid-point of these excursions noted, and the platform be brought to a stop at this point, or electrical damping can be inserted in the system to damp out this oscillation leaving the platform correctly aligned.

It should be noted that τ increases without limit as θ approaches 90°. The system fails at the poles and becomes increasingly sluggish as the

poles are approached. This is a basic limitation on gyrocompass alignment.

An alternative approach to the use of the Earth rate vector as an azimuth reference comes through considering the platform as an angular velocity resolver. In this case the platform is aligned approximately in azimuth, and leveled accurately using gravity as a reference. The computer applies Earth rates about the x and z axes computed from the known position of the inertial navigator. If the platform is correctly aligned, it will remain fixed in orientation with respect to the Earth's surface. However, if as shown in Fig. 26*a*, there is an initial azimuth

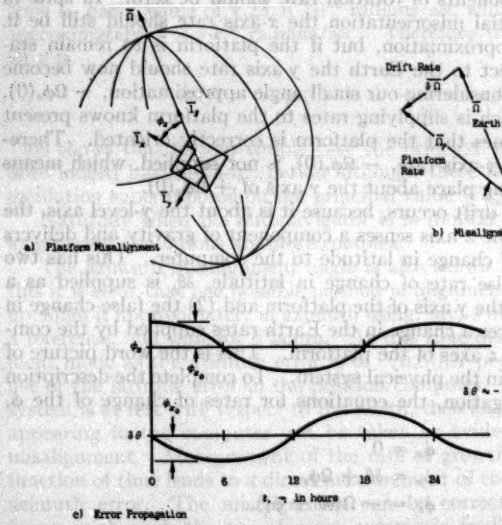


FIG. 26. Inertial navigator gyrocompass alignment.

misalignment angle ϕ_z, then, as shown in Fig. 26*b*, the rate supplied to the platform does not coincide with Earth rate. Therefore, there will be an error in the platform rate $\delta\bar{\Omega}$. This means that the platform will drift with respect to the Earth's surface at this rate.

Take the simple case of a platform misaligned in azimuth on the equator with its x axis pointing approximately north, its y axis pointing approximately west, and its z axis approximately vertically upward. Now define a set of angles ϕ_x, ϕ_y, and ϕ_z to be the small misalignment angles of the platform axes from the position they would occupy for a perfect platform correctly aligned. These angles are assumed to

be sufficiently small so the cosine of any angle equals 1 and the sine equals the radian measure of the angle itself. With this restriction the angles can be treated as vectors while the platform is rotating without the necessity of using a set of Euler angles to represent the platform misorientation. For this example the initial values of the ϕ's are $\phi_{x0} = 0$, $\phi_{y0} = 0$, and $\phi_{z0} = \phi_z(0)$.

The equations for the ϕ angle propagation can be readily set up realizing that the platform can be regarded as a three-dimensional angular rate resolver. For this example, with the platform correctly oriented, full Earth rate, Ω, should be present about the x axis while the y and z axis components of rotation rate should be zero. In spite of the platform's initial misorientation the x-axis rate should still be Ω, to a first order approximation, but if the platform is to remain stationary with respect to the Earth the y-axis rate should now become $-\Omega \sin \phi_z(0)$, or considering our small angle approximation, $-\Omega\phi_z(0)$. The computer which is supplying rates to the platform knows present position and assumes that the platform is correctly oriented. Therefore, this spurious y-axis rate, $-\Omega\phi_z(0)$, is not supplied, which means that a drift is taking place about the y axis of $+\Omega\phi_z(0)$.

As soon as this drift occurs, because it is about the y-level axis, the accelerometer on the x axis senses a component of gravity and delivers false indications of change in latitude to the computer. This has two effects: (1) the false rate of change in latitude, $\dot{\delta\theta}$, is supplied as a torquing signal to the y axis of the platform and (2) the false change in latitude, $\delta\theta$, produces a change in the Earth rates supplied by the computer to the x and z axes of the platform. This is the word picture of what is happening in the physical system. To complete the description of the error propagation, the equations for rates of change of the ϕ_i angles are

$$\begin{aligned} \dot{\phi}_x &= 0 \\ \dot{\phi}_y &= \dot{\delta\theta} + \Omega\phi_z \\ \dot{\phi}_z &= -\Omega(\delta\theta + \phi_y). \end{aligned}$$

The change in indicated latitude, $\delta\theta$, is due entirely, in this simple example, to drift about the y axis and therefore due to the double integration of the instantaneous angular accelerations $-\phi_y \frac{g}{a}$. Taking the Laplace transformation and making the obvious substitutions gives the following form for the equations:

$$s\phi_x = 0$$

$$s\phi_y + \frac{g}{as}\phi_y - \Omega\phi_z = 0$$

$$\Omega\left(\frac{g}{as^2}\phi_y + \phi_y\right) + s\phi_z = \phi_z(0).$$

Taking the determinant of the left hand side of these equations gives

$$\frac{\left(s^2 + \frac{g}{a}\right)(s^2 + \Omega^2)}{s}; \quad \text{set } \frac{g}{a} = \omega_0^2.$$

Solving for $\delta\theta$, the latitude error, under those conditions in which the sole driving function is the initial value of $\phi_s = \phi_s(0)$ gives

$$\delta\theta(t) = \frac{-\omega_0^2\Omega\phi_s(0)}{(\omega_0^2 - \Omega^2)}\left(\frac{\sin \Omega t}{\Omega} - \frac{\sin \omega_0 t}{\omega_0}\right).$$

Now ω_0 has a value of 1.23×10^{-3} radians/sec while Ω has a value of approximately 7×10^{-5} radians/sec. Therefore the first order form for θ is

$$\delta\theta(t) = -\phi_s(0) \sin \Omega t - \frac{\Omega}{\omega_0} \sin \omega_0 t.$$

This shows that for our simple case the principal component of error does indeed have a 24-hr. period although there is an 84-min. period oscillation superimposed on the principal value (Fig. 26*c*).

Naturally when the autonavigator is in motion the expressions for long period oscillations are more complex.

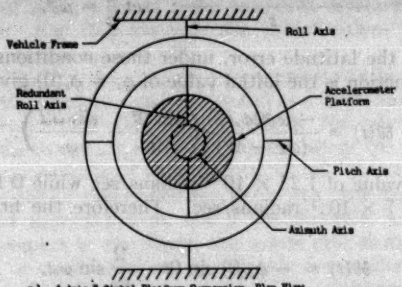
The azimuth misalignment angle is also shown in Fig. 26*c*. From this it is clear that one possible method of alignment is: set the system up, align it approximately in azimuth using, say, a magnetic compass as a reference, and wait for 6 hours. At this time discontinue the alignment mode as the azimuth error is 0. Another possibility is to make use of the latitude error in the position computer. Knowing that the system is at rest with respect to the Earth, then any change in latitude appearing in the computer can be taken as evidence of an azimuth misalignment. Measurement of the rate of growth of this error as a function of time leads to a direct measurement of the magnitude of this azimuth error. The misalignment can be corrected either by step input or through the use of some automatic feedback system incorporating damping. However, although it did not appear in the equations for ϕ_s in this case (on the equator $\cos\theta = 1$) $\cos\theta$ does enter the picture, so this system of azimuth alignment fails at the poles.

In this way using both the Earth's gravitation and the Earth's rotation as reference, combined with a knowledge of present position, it is possible to make the alignment of the inertial navigator a completely self-contained operation.

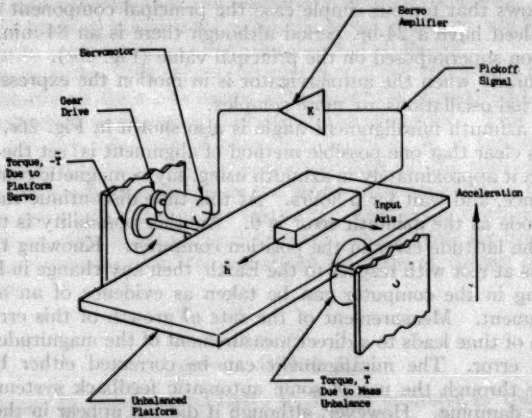
3. *Stabilization*

In our system pictures to date, we have shown physical connections only between the gyro package and the accelerometer platform. In

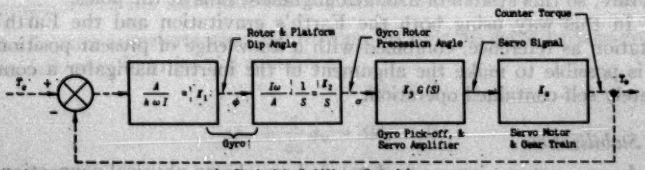
an actual vehicle, additional gimbals are required to isolate these systems from vehicle motion. A possible gimbal system is a 3-axis 2-gimbal system with roll being the outer axis, pitch the intermediate



a) 4-Axis 3-Gimbal Platform Suspension, Plan View



b) Single-Axis Stabilization



c) Single Axis Stabilizing Control Loop

FIG. 27. Platform stabilization.

axis, and azimuth the inner axis. The roll axis is connected directly to the vehicle frame while the accelerometers and gyros are mounted on the inner table. If additional gimballing is required between the accelerometers and the gyros, as for example in a geometric system, then these would have to be added with a resultant increase in dimensions and weight of the external gimbals.

If complete maneuverability is required, one additional gimbal must be added, (Fig. 27*a*) making it a 4-axis 3-gimbal system, in order to avoid the problem of gimbal lock. Usually this axis will be a redundant roll axis or a redundant pitch axis depending on the mounting employed, with a limited degree of freedom.

The mechanization for stabilization is illustrated in Fig. 27*b*. Here stabilization about one axis is being considered, the platform as shown is obviously unbalanced. An acceleration in the direction indicated would result in a torque on the platform tending to cause it to rotate. This is sensed by the control gyroscope. The torque causes a small precession of the gyro rotor. The resulting pick-off angle generates an error signal. This is fed through an amplifier to a torque motor mounted between gimbal and control axis for the platform. The sign of the error signal is such that the torque applied to the platform through the torque motor is in the opposite direction to the torque due to mass unbalance. If the system is correctly adjusted, then virtually equal but opposite torques will be applied to the platform from these two sources, mass unbalance and the torque motor, with the result that the platform will remain stationary.

The error sensing and feedback loop for this single axis stabilization is shown in Fig. 27*c*. As can be seen, the relation between the disturbing torque T_y and the resultant platform error angle ϕ_y is given for $G(s) = 1$ by

$$\phi_y(s) = T_y(s) \frac{sK_1}{s + K_1K_2K_3K_4}$$

where K_3 is the amplifier stage gain, K_1 and K_2 are constants relating mechanical properties of the control gyro, and K_4 is the conversion factor from servo amplifier signal through servo motor and gear train to counter torque.

The system as shown is stable, with errors damped exponentially. However, it should be noted that there is already one integration between the disturbing torque and the error signal, giving a 90° phase lag. The characteristics of practical system elements must be considered, such as back-lash in the gear train between the torque motor and the control axis, transfer functions of the various amplifiers and transmission paths in the system. It is possible to obtain sufficient phase shift at some frequencies of oscillatory disturbing torques to cause the

system to amplify these oscillations leading to "tumbling" of the platform. For this reason, the design of the platform torquer system requires a detailed servo analysis. It must be tailored to the platform to be controlled, considering the environment in which it is to exist. Skillful use of practical components in building up $G(s)$ leads to a workable system in most cases.

The complete problem of platform stabilization is somewhat more complex than that shown for the single axis. Error signals sensed by gyros on the platform must be resolved through an azimuth rotation angle ψ to the roll and pitch axis of the gimbal system. This is illustrated in Fig. 28*a*. As the vehicle heading changes ψ will usually

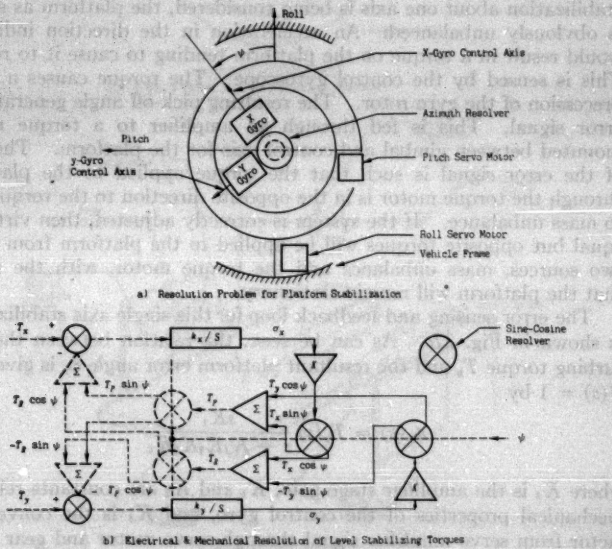


FIG. 28. Two-axis platform stabilization.

change. For example, in a latitude-longitude system the x-gyro control axis will always point north, whereas the roll gimbal axis is along the center-line of the vehicle, so the angle between the x-gyro axis and the roll gimbal axis depends on present position and vehicle heading. The outputs of both the x and y gyros then must be resolved through the angle ψ to determine the error signal being amplified and applied to the roll and pitch gimbal torquers. In block form this is shown in Fig. 28*b*,

where the electrical part of the operation is shown in solid lines with the mechanical part in dotted lines.

As an additional illustration of the difficulties in platform stabilization I would like to mention the problem of cross coupling and its effect on stabilizing gyros.[3] If oscillatory inputs ϕ_x and ϕ_y about the x and y axes, respectively, are in phase there will be a resultant average torque experienced by the gyro controlling about the x axis. This causes the gyro to drift, producing a system error even though the gyro itself is working properly. In addition the stabilization system is keeping the average value of $\dot{\sigma}$, the gyro precession rate, equal to 0 as it should.

B. SUBSYSTEMS

1. *Inertial Components*

a. Gyroscopes

Gyroscopes employed as directional references for inertial navigation systems must of necessity be several orders of magnitude better than the gyros previously used for attitude references in aircraft. A drift rate of 1° per hour, for example, while very satisfactory for an attitude reference in a low speed aircraft, would contribute an error of 60 nautical miles per hour in an inertial navigation system. This is not a reasonable degree of accuracy. Random drift rates of the order of 0.02° per hour have been quoted for unclassified HIG gyros.[4]

Many of the current inertial systems are based on the use of three single-degree-of-freedom gyros as directional references. The method of use of a gyro as a control for a single axis is shown in Fig. 29. Figure 29*a* shows in pictorial form the gyroscope itself. A rotor *a* spins under torque from a drive motor *b* in an inner gimbal ring *d*. This gimbal in turn is supported by the output axis bearings *j* which will be of a very low coercion type. Any precession of the rotor with respect to the case *g* is sensed by the pick-off *b* which produces a signal sent to the amplifier and servo control motor for the controlled platform axis.

The complete control path is shown in Fig. 29*b*. As can be seen from this diagram, if the direction of an external torque, T, is reversed, T_1, then the gyro precession rate, $\dot{\sigma}$, changes direction. The pick-off is so arranged that the direction of precession of the gyro rotor with respect to the case is sensed as well as the magnitude of this angle. Therefore, a counter-balancing torque can be generated through the servo motor, correct both in magnitude and sign, to maintain correct platform orientation.

[3] This is covered in detail in a paper by R. H. Cannon published in the December issue of the A.S.M.E., 1957.

[4] Philip Klass, in *Aviation Week*, see reference 4 in the Selected Bibliography appended to this paper.

Details of construction of any particular gyroscope for the most part are classified, but some general remarks about desirable characteristics can be made. For one thing the angular momentum of the rotor should be high in order to keep the gyro drift rate, ϵ, small in the face of the inevitable error torques. In addition this high angular momentum produces maximum restoring force on the platform in the presence of an external disturbing torque. This requirement is less stringent when high performance platform servos are used, but even here for high-frequency disturbances much of the stabilizing torque is supplied by the gyro.

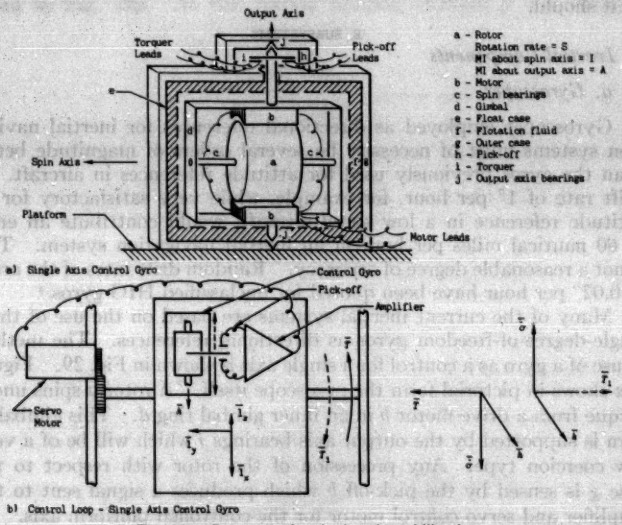


FIG. 29. Gyroscope for single-axis stabilization.

High angular momentum can be obtained either by using a large mass in the rotor or a high spin rate. If the mass is large, the problems of structural rigidity, heating due to higher power input requirements, and bearing deformation under accelerations all require careful attention, while if the spin rate is very high, the problem of long-life spin bearings becomes acute. Some reasonable compromise is usually made between these conflicting methods of obtaining the necessary angular momentum.

The output axis bearing must be of a low coercion type, for if the gyroscope is to be used to control about only one input axis, it will be so arranged that the torques about this output axis are forced to 0. Any

residual torque causes precession of the rotor, which is then interpreted as being due to an external torque applied to the platform. A compensating torque will be generated through the servo amplifier and torquer motor causing platform drift.

In order to put the least possible load on the bearing, the gyro rotor and inner gimbal are often surrounded by a sealed case and floated in a flotation fluid as shown in Fig. 29*a*. If the displaced mass of flotation fluid is equal to the mass of the gyro wheel plus flotation case, then neutral buoyancy has been achieved and the load on the output axis bearing has been reduced to a minimum. An additional feature of the flotation fluid is that if its viscosity is sufficiently high it provides some natural damping for oscillations between the gyro float and the gyro outer case.

While the gyroscope shown in Fig. 29*a* could not control platform rotation about its spin axis it could control platform rotation about two axes perpendicular to each other and to the spin axis, with the addition of another gimbal. Used in this manner two two-axis gyros are sufficient to stabilize a platform.

In the arrangement shown only one axis is controlled. If a torque occurs about the output axis the case moves freely with respect to the gyro wheel. This movement is sensed by the pick-off and a torque is applied through the gimbal servo motor to the platform. There is no torque being applied externally to the platform to counter, so under the influence of this torque the gyro precesses (Fig. 30). In this fashion,

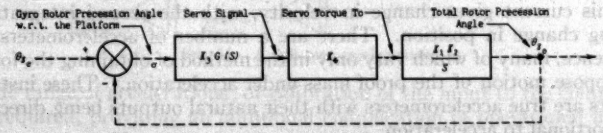


FIG. 30. Following of azimuth command angle by level control gyro.

when the platform moves in azimuth under control of the azimuth gyro, the level gyro automatically follows this rotation, while still controlling the platform in level.

There are a number of sources of gyro drift present in the gyroscope shown in Fig. 29*a*. Any action which causes a torque to be developed between the case and the gyro wheel can result in gyro and platform drift. For example, the lead-in wires to the gyro motor may exert small but significant torques causing appreciable drift rates. If the rotor is not perfectly symmetrical with respect to the center of buoyancy then any acceleration acting on this mass unbalance causes a torque to act on the rotor with resultant gyro drift. Convection currents in the flotation fluid also cause drift-producing torques between the outer case and the float.

In addition to mass unbalance there is the problem of anisoelastic deformation of the spin bearing under accelerations causing the center of mass of the gyro wheel to become displaced from the direct line of action of the acceleration and the center of buoyancy, giving another source of disturbing gyro torques and consequent gyro drift.

These are but a few of the many possible sources of extraneous torques. From this brief description it can be seen that the design of a gyroscope to meet high performance specifications is a very difficult engineering task.

b. Accelerometers

Accelerometers in general rely on the fact that a force is necessary to change the linear momentum of any mass. The mass is so arranged that any force acting along a particular axis, its sensitive or input axis, is measurable. Knowing the force and the mass, acceleration can be calculated from Force = mass × acceleration. A possible accelerometer configuration is shown in Fig. 31. Here a proof mass is suspended from a pivot point. With no acceleration occurring along its input axis the pick-off is held in a neutral position. When acceleration occurs, the proof mass tends to swing to the right or left, depending on the sign of the acceleration, and an output signal is generated, amplified, and fed back through the torquer coils to maintain the proof mass in its central position. The magnitude of the torquing current required is a direct measure of the acceleration taking place. A single integration of this current gives change in velocity, with the second integration giving change in position. There are a number of accelerometers in existence, many of which vary only in the method of obtaining the force to oppose motion of the proof mass under acceleration. These instruments are true accelerometers with their natural outputs being directly proportional to acceleration.

A different class of accelerometers can also be obtained in which the natural output is directly proportional to velocity, the first integral of acceleration. As has been stated in the preceding section, mass unbalance makes a gyroscope sensitive to accelerations. A proof mass can be placed on the inner gimbal in such a fashion that only acceleration along one axis causes the gyro rotor to precess, with torques due to accelerations along other axes being automatically counteracted by the system. Further, it has been stated that a torque applied to a gyroscope determines the precession rate of the rotor. Therefore, the total precession angle is equal to the integral of the applied torque, which is in turn directly proportional to the acceleration occurring along the sensitive axis. In this case then, a measure of the precession angle of this gyroscope gives the accumulated velocity of the system.

Consider a different instrument in which the motion of the proof

mass had been opposed by a viscous drag. The displacement of the proof mass from its initial position is a measure of system velocity in the direction of its sensitive axis. It is also possible to produce a doubly integrating accelerometer with a natural output equal to the total position change of the system.

In general, the problems associated with accelerometers are easily determined. The scale factor relating usable output to acceleration must be linear over very wide ranges of acceleration, and constant in time. The physical construction of the instrument must not introduce false accelerations into the system.

As an example consider the accelerometer shown in Fig. 31. Torquer linearity must be maintained for an electromagnetic torquer over a

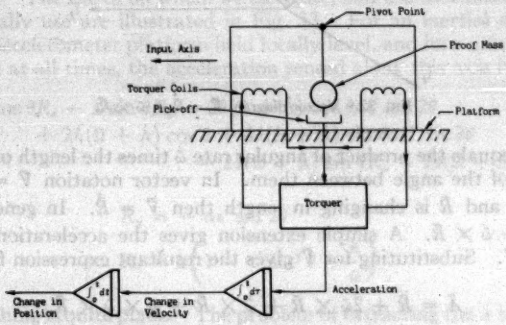


FIG. 31. A null-type accelerometer.

wide range of accelerations; or calibration of the input to the computer is required, providing an additional complication in operational use of the system. Also there is the matter of maintaining stability of this torquer scale factor over wide ranges of operating environmental conditions. In this instrument there would be problems such as friction in the pivot bearings, biases due to physical placement of the pick-offs, and shift in this placement under vibration and acceleration, and all of the ills of physical instruments. In addition, with this type of accelerometer, the problems of drift and bias settings for the necessary integrators used to convert its output to velocity and position changes have to be solved.

2. *Computer Characteristics Important to Inertial Guidance*

Before examining the characteristics of computers used in inertial navigation systems it is advisable to examine the equations to be mechanized. Their contents will determine the desirable computer characteristics.

In general the motion of a vehicle can be visualized easily by following the tip of the radius vector $\bar{R}$ drawn from the origin of a coordinate system, (in our case usually the center of the Earth) and rotating at an angular rate $\bar{\omega}$ (Fig. 32). If the length of $\bar{R}$ is constant, vehicle

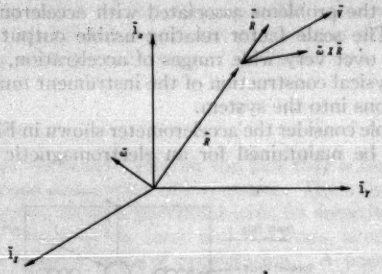


FIG. 32. Vector diagram $\bar{V} = \dot{\bar{R}} + \bar{\omega} \times \bar{R}$.

velocity equals the product of angular rate $\bar{\omega}$ times the length of $\bar{R}$ times the sine of the angle between them. In vector notation $\bar{V} = \bar{\omega} \times \bar{R}$. If $\bar{\omega} = 0$ and $\bar{R}$ is changing in length then $\bar{V} = \dot{\bar{R}}$. In general then $\bar{V} = \dot{\bar{R}} + \bar{\omega} \times \bar{R}$. A simple extension gives the acceleration $\bar{A} = \dot{\bar{V}} + \bar{\omega} \times \bar{V}$. Substituting for $\bar{V}$ gives the resultant expression for $\bar{A}$

$$\bar{A} = \ddot{\bar{R}} + 2\bar{\omega} \times \dot{\bar{R}} + \dot{\bar{\omega}} \times \bar{R} + \bar{\omega} \times \bar{\omega} \times \bar{R}. \qquad (1)$$

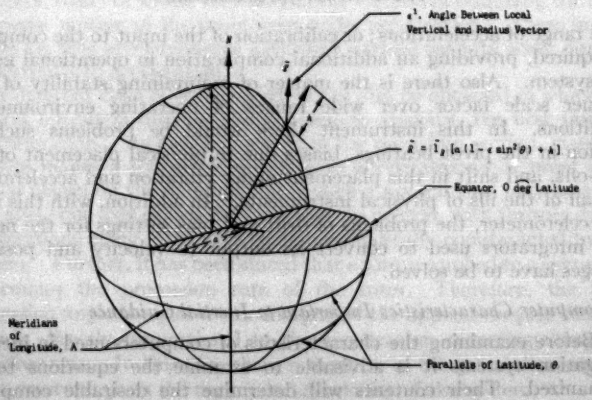

ϵ^1, Angle Between Local Vertical and Radius Vector
$\bar{R} = \bar{i}_R [a(1 - \epsilon \sin^2\theta) + h]$
Equator, 0 deg Latitude
Meridians of Longitude, λ
Parallels of Latitude, θ

FIG. 33. The Earth as an ellipsoid of revolution.

We can expect that the Earth rate must be considered when navigating on the Earth. Indeed it appears in Eq. 1 as part of $\tilde{\omega}$, the other part being plane's rate $= \frac{\bar{V}}{R}$. Are we justified in stopping with the Earth's spin? Why not consider the rotation of the Earth about the sun? A moment's thought reminds us that the Earth moves around the sun at satellite velocity. This means that centripetal acceleration and mass attraction are in balance and so can be ignored.

One final factor which must be considered if accurate inertial navigation is attempted, is the Earth's ellipticity. The Earth is usually represented by an ellipsoid of revolution, allowing for the fact that the polar diameter is some 23 nautical miles shorter than the equatorial diameter. The Earth on which we navigate, and the coordinate system we normally use are illustrated in Fig. 33. For an inertial navigator with its accelerometer platform held locally level, and its x axis pointing east-west at all times, the acceleration sensed along this axis is

$$A_x = \ddot{\Lambda} \cos \theta R_z - \ddot{\Lambda} \sin \theta R_y - 2\dot{\theta}(\Omega + \dot{\Lambda}) \cos \theta a\epsilon' \sin 2\theta + 2\dot{h}(\Omega + \dot{\Lambda}) \cos \theta + 4\dot{\theta}(\Omega + \dot{\Lambda}) \sin \theta a\epsilon' \cos 2\theta - 2\dot{\theta}(\Omega + \dot{\Lambda}) \sin \theta R_z - 2\dot{\theta}(\Omega + \dot{\Lambda}) \cos \theta R_y$$

where

$$R_y \approx -a\epsilon' \sin 2\theta$$

$$R_z \approx a\left(1 - \epsilon' \sin^2 \theta + \frac{h}{a}\right).$$

One thing is quite plain. The problem of extracting the $\ddot{\Lambda}$ term from this expression for A_x is not trivial. Another apparent fact is that a large number of derivatives must be taken, and that sines and cosines must be generated in quantities in solving the navigation problem. These factors should be borne in mind in determining the suitability of a particular computer for real-time inertial navigation.

No attempt will be made here to go into the details of computer mechanization. Only those characteristics of analog and digital computers which have a bearing on their use in inertial guidance systems are presented for a basis of comparison.

a. Analog Computers

1. Most instruments naturally produce analog outputs, therefore the tie-in problems between analog computers and inertial guidance instruments are usually simple. For example, the outputs of the pick-offs and the inputs to the torquers in the accelerometer and gyro shown in this paper are analog in form.

2. Many trigonometric functions are mechanized in the solution of inertial guidance problems. In an analog computer, the use of sine-

cosine potentiometers to produce these functions offers a simple solution to their formation. In addition resolvers can be used readily to resolve functions through angles corresponding to physical angles in the system.

3. It is difficult, although not impossible, to obtain an accuracy greater than 1 part in 1000 from an analog computer. Indeed, under many conditions it is difficult to obtain even this accuracy.

4. For precision in analog computation you pay with precision in the physical components used for computing. This poses a difficult fabrication problem.

5. Analog computer components can be made with much lower moments of inertia than the physical system being controlled. Therefore, real-time computation for control of physical systems is almost always possible using an analog computer.

b. Digital Computers

1. A digital computer usually requires analog-to-digital converters when used with inertial guidance instruments.

2. Very high computational accuracy can be obtained readily without requiring corresponding accuracy in the fabrication of component parts.

3. It may be difficult to keep up with a real-time control problem.

4. Trigonometric functions are generated by calculating terms of a series or some similar device unless digital differential analyzer techniques are employed; therefore, the calculation of these functions, highly prevalent in the solution of inertial guidance problems, can be quite time consuming.

5. Incremental or whole number inputs may be acceptable although this depends on the particular computer.

6. For control, the over-all system becomes a sampled-data system with the digital computer in the control loop; therefore, stability determination becomes a more complex problem than for an all-analog system.

7. A basic arithmetic unit must be supplied without regard to the total amount of computation to be done. Therefore, for very simple computing problems a digital computer may be at a disadvantage by comparison with an analog computer in weight, volume, and simplicity.

F. ANALYSIS OF ERROR CONTRIBUTIONS

1. Bias

Using the concept of an inertial guidance channel as a position servo, the effect of a bias error in the accelerometer of K ft/sec^2 is easily determined. The error equations are

$$\delta\ddot{\theta}_y + \omega_0^2\delta\theta_y = \frac{K}{a}$$

$$\delta\Theta_y(s) = \frac{K/a}{s(s^2 + \omega_0^2)}$$

$$a\delta\theta_y(t) = \frac{K}{\omega_0^2}(1 - \cos\omega_0 t).$$

For $K = \frac{1}{100}$ ft/sec^2; $a\delta\theta_y(t) = 6{,}600\ (1 - \cos\omega_0 t)$ ft. The resultant error is shown in Fig. 34 with both the position error and the platform

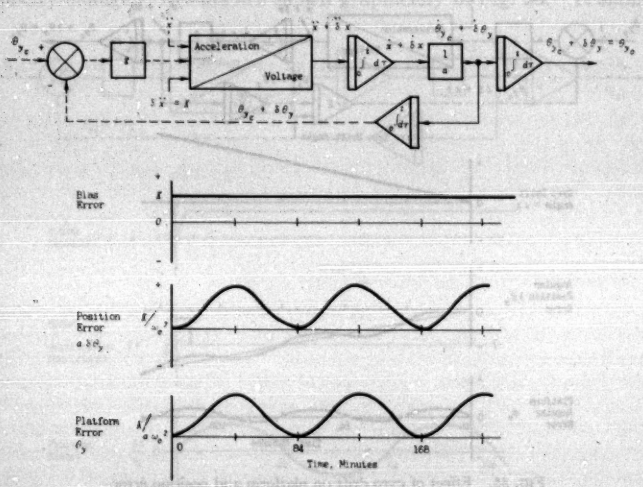


FIG. 34. Effect of accelerometer bias shift on position error.

angle error sketched in. It can be seen that the position error is of the form $(1 - \cos\omega_0 t)$, and the resultant platform error angle follows the same form.

2. Gyro Drift

Using a constant gyro drift rate, ϵ, and ignoring the angular rate coupling which takes place over a long term in the actual guidance platform, the relation between drift rate and position error is determined as follows:

$$\delta\ddot{\theta}_y + \omega_0^2\delta\theta_y = \frac{\epsilon g}{a}$$

or

$$\delta\Theta_y(s) = \frac{-\epsilon g/a}{s^2(s^2 + \omega_0^2)};$$

so

$$a\delta\theta_y(t) = -a\epsilon\left(t - \frac{1}{\omega_0}\sin\omega_0 t\right).$$

After 6 hours a gyro drift rate of 0.1°/hr produces a 36-nautical mile position error. The resultant errors for both position and platform angle are shown in Fig. 35.

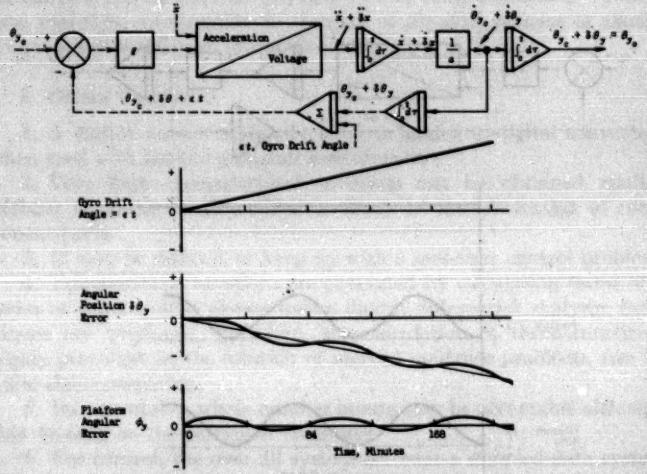


FIG. 35. Effect of gyro drift on platform and position error.

There is an interesting difference between the results for this constant gyro drift rate and for a bias error in the accelerometer. Here the platform angle oscillates sinusoidally about a 0 value, although the position error is building up with time. The reason is that gyro drift is an active driving source in the platform system. On the average this drift is cancelled by the incorrect torquing of the platform due to the position error. This leaves only the 84-min. period component of error affecting the platform angle. If the platform is being used as a vertical reference, then it is possible to obtain good accuracy in the indicated vertical in spite of large errors in computed position. The 0.1°/hr. gyro drift rate for example causes peak platform oscillations of only 1.4 min. of arc.

3. *Impulse*

The effect of an impulse acceleration with an area k ft. per second applied to the system can be calculated as follows: $\delta\ddot{\theta}_y + \omega_0^2\delta\theta_y = \dfrac{k\delta(t)}{a}$; where $\delta(t)$ is Dirac delta function; or $\delta\Theta_y(s) = \dfrac{k/a}{s^2 + \omega_0^2}$; so $a\delta\theta_y(t) = \dfrac{k}{\omega_0}\sin\omega_0 t$. For a scale factor error of 0.1 per cent in an accelerometer used to measure an abrupt change in velocity of 1000 ft./sec., the resultant position error $a\delta\theta_y(t) = 800 \sin \omega_0 t$ ft. with $t = 0$ at the time of velocity change. Results are shown graphically in Fig. 36. It should

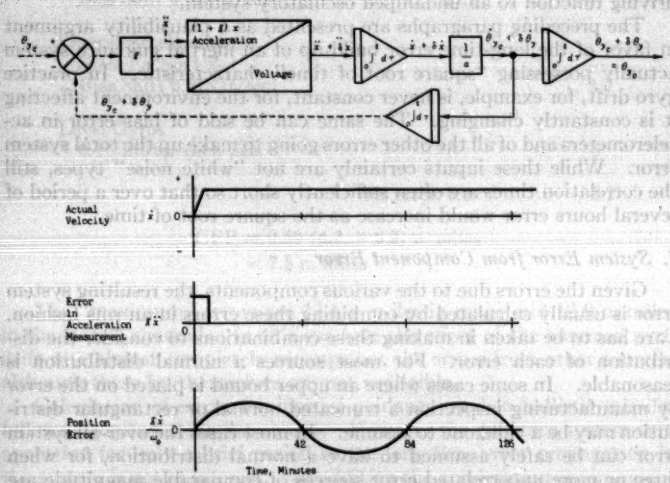


FIG. 36. Position error due to pseudo-impulse acceleration error.

be observed that due to the long period of the system, disturbances occurring over a 3- to 5-min. time interval can actually be treated as impulses occurring at the center of this interval with very little error.

4. *Noise*

A series of disturbing impulses of random magnitude and sign applied to the guidance system generate a series of sine waves with random phase differences. As has been stated previously disturbances of up to 5 minutes duration affect the system almost as impulses. The problem of determining system position error for such an input then resembles

the "random walk" problem in probability theory. Here n steps are taken along a line from an origin, the rms value of each step being K ft., with equal probability that each step is forward or backwards. The rms value of displacement from the origin after n steps, for n sufficiently large, is approximately $K\sqrt{n}$ ft.

Similarly for the inertial guidance system if the rms value of errors due to individual disturbing "impulses" is K_1 feet, then after n of these impulses, for n sufficiently large, the rms value of position error should be $K_1\sqrt{n}$ ft. On the assumption that these noise impulses are uniformly distributed in time, the resultant position error increases as $K_2\sqrt{t}$. This result is in agreement with the form of error build-up found when "white" noise, that is noise with a zero correlation time, is applied as a driving function to an undamped oscillatory system.

The preceding paragraphs are presented as a plausibility argument in favor of the long-time error build-up of an inertial guidance system actually possessing "square root of time" characteristic. In practice gyro drift, for example, is never constant, for the environment affecting it is constantly changing. The same can be said of bias error in accelerometers and of all the other errors going to make up the total system error. While these inputs certainly are not "white noise" types, still the correlation times are often sufficiently short so that over a period of several hours error would increase as the square root of time.

5. System Error from Component Error

Given the errors due to the various components, the resulting system error is usually calculated by combining these errors in an rms fashion. Care has to be taken in making these combinations to consider the distribution of each error. For most sources a normal distribution is reasonable. In some cases where an upper bound is placed on the error by manufacturing inspection a truncated normal or rectangular distribution may be a valid one to assume. In most cases the over-all system error can be safely assumed to have a normal distribution, for when three or more uncorrelated error sources of comparable magnitude are combined the resultant error distribution is a close approximation to the normal.

Two parameters often quoted as measures of system performance can be derived from the rms value of system error. One is the circular probable error; that is, the radius of a circle inside which 50 per cent of the system errors would fall. The relation between the rms errors in each channel of the navigator and this circular probable error is given approximately by

$$\text{CEP} = 0.589[(\text{Range Error})_{\text{rms}} + (\text{Lateral Error})_{\text{rms}}],$$

a slight overestimate. The other parameter often quoted is a "so-called" maximum error; an error which is greater than 95 per cent of the

position errors occurring. For a normal distribution of errors, this is twice the rms error previously calculated.

For example, assume the following distribution of component errors for an inertial guidance system used in a Mach 1 vehicle on a 1-hr. great-circle flight.

TABLE I.

(rms values used throughout unless otherwise noted)

Component Error Source	Range Error, nautical miles	Lateral Error, nautical miles
Range Accelerometer		
Scale Factor 1%	1.3	—
Bias Shift 0.015 ft/sec²	1.7	
Lateral Accelerometer		
Scale Factor 1%	—	—
Bias Shift 0.015 ft/sec²		1.7
Level Gyros		
0.1°/hr. for each axis	6	6
rms error	6.3	6.2

$$\text{CEP} \approx 0.59\,(6.3 + 6.2) \text{ n. miles}$$
$$\approx 7.5 \text{ n. miles}$$

The "maximum" errors are 12.6 n. miles in range and 12.9 n. miles in the lateral channel. In this example the effect of the assumed gyro drift rate almost completely swamps out the errors from other sources, although these by themselves are not negligible.

Calculation of errors for inertial guidance used in ballistic missiles follows a somewhat similar course. However the guidance system is in operation for only a brief period so the long-time error propagation in the system is not of interest. In addition the acceleration environment is very severe. However, given instrument characteristics and the flight profile, guidance errors at power cut-off can be determined. The manner in which these errors affect the impact point can also be calculated. Given this information navigation errors can be computed in a straight-forward fashion.

For example, consider a ballistic mail delivery service from Los Angeles to Honolulu—an airline distance of about 2500 miles.

Then for an azimuth misalignment error, ϕ_z, of 2 milliradians rms, the resultant lateral error is given approximately by $\phi_z \times$ Range, in this case $\frac{2}{1000} \times 2500$ miles, or 5 miles.

If the average horizontal velocity is 5000 miles/hr., then flight time

is $\frac{1}{2}$ hr. An error of 0.2 per cent rms in horizontal velocity determination then leads to a range error of $5000 \times \frac{2}{1000} \times \frac{1}{2} = 5$ miles in range.

If these are the only significant errors, then

$$\text{system CEP becomes} = 0.59(5 + 5) \text{ miles}$$
$$= 6 \text{ miles}$$

With this system time saved from post-office Los Angeles to post-office Honolulu over more routine forms of delivery is difficult to calculate.

G. AIDED INERTIAL SYSTEMS

It is possible to combine information from other navigation devices with an inertial guidance system to produce an output better in some ways than the information available from either of the original sources. Methods of combining this information with an inertial guidance system follow easily if each channel of the guidance system is considered as a positional servo. External information for example might be available in the form of position information or velocity information. These possibilities are considered separately.

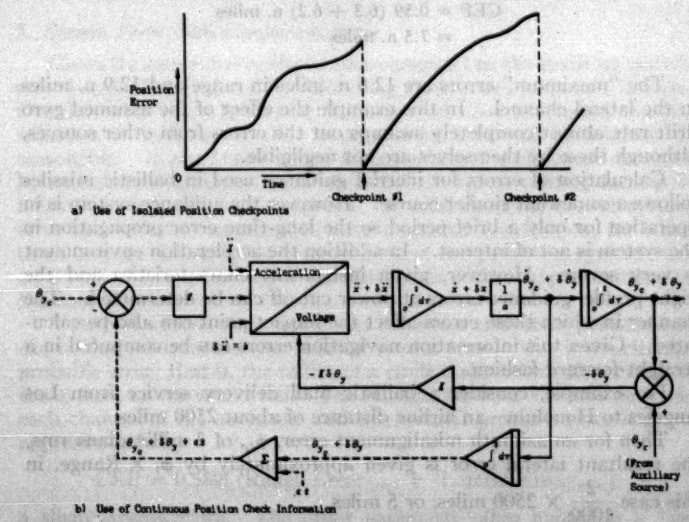


FIG. 37. Use of auxiliary position information for inertial navigators.

1. Position Information

a. Discrete

If position information is available only at intervals then it can be employed to reduce the position error of the inertial guidance system to conform with these discrete position checks. The result of using such information is shown in Fig. 37*a*. It is observed that the position error build-up after a checkpoint is more rapid than the initial build-up. This is due to the velocity error which has built up during the initial period of operation, T, and has not been reduced by the use of a single position check.

b. Continuous Position Information

If a system such as Loran is available to give a continuous check on position for some parts of the vehicle's journey, it is possible to combine the output of the navigator with this position information in a manner shown in Fig. 37*b*. Using the revised feedback, shown in Fig. 37*b*, the natural period of the system is decreased and the magnitude of position errors resulting from a number of driving sources has been reduced. The system transfer function has become $\frac{1}{s^2 + K/a + \omega_0^2}$, so for $K = 9g$ an accelerometer bias change of 0.015 ft/sec² now gives an error of 1000 $(1 - \cos \sqrt{10}\omega_0 t)$ ft.

The errors due to gyro drift are also reduced.

$$-(\delta\theta_v + \epsilon t)g - K\delta\theta_v = a\delta\ddot{\theta}_v \text{ and set } K = kg$$

or

$$a\delta\ddot{\theta}_v + \delta\theta_v(k+1)g = -\epsilon g$$

or

$$\delta\ddot{\theta}_v + (k+1)\omega_0^2\delta\theta_v = -\epsilon\omega_0^2$$

or

$$\delta\theta_v = \frac{-\epsilon}{(k+1)}\left[t - \frac{1}{\sqrt{k+1}\omega_0}\sin\sqrt{k+1}\omega_0 t\right]$$

$$= \frac{1}{k+1} \times \text{(Position Error in Unaided Inertial System)}$$

From the results of the preceding analysis there may be some question as to why an inertial navigator would be used at all with this position information available. The answer will depend on the characteristics of the position signal being compared with the inertial navigator output. In the cases outlined so far the aiding position information was assumed to be perfect. In practice this rarely happens. For example the position information may be accurate on the average, but vary

suddenly from moment to moment. Differentiating this information provides only a poor source of instantaneous velocity information. By first using the smoothing provided by the inertial system a signal which can be differentiated to give accurate instantaneous velocity can be generated. In addition the inertial system can extrapolate beyond areas of Loran or similar range-limited coverage, or fill in gaps in this coverage.

2. Velocity Damping

If an external source of velocity information is available, such as airspeed or Doppler radar, then a comparison can be made between this source and the velocity supplied by the inertial guidance system. The resultant error signal, multiplied by some suitable transfer function, can be fed back to the input of each inertial navigator channel, as shown in Fig. 38. Here in its simplest form $G(s)$ might be equal to

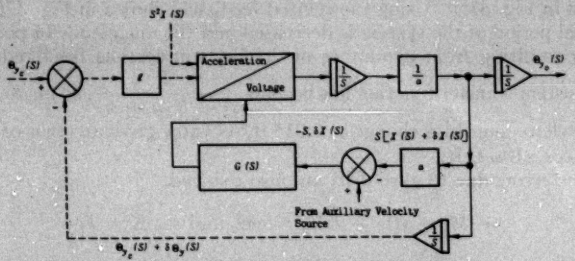


FIG. 38. Velocity damping of inertial guidance.

$-k$. In that case a simple second order system is formed with the transfer function $\dfrac{1}{s^2 + ks + \omega_0^2}$. Disturbances are damped out to their average value by the factor $e^{-kt}/2$.

One disadvantage of this system is that a constant reference velocity error results in a constant position error after the oscillatory term has been damped out. This can be eliminated by the use of a third order system where $G(s)$ becomes $-\dfrac{k_1 s}{s + k_2}$. The system transfer function becomes $\dfrac{s + k_2}{s^3 + (k_1 + k_2)s^2 + \omega_0^2 s + k_2\omega_0^2}$. In this third order system position error due to a constant reference velocity error is damped to 0. However, sudden shifts in this error cause transient position errors which take time to damp out.

It is tempting to consider using an external measurement of ground speed, perhaps Doppler radar capable of good average velocity accuracy, to reduce the dependence of the combined systems on gyro drift rate. However, azimuth gyro drift cannot be controlled in this fashion, and for extended operating times becomes a dominant error source.

The actual form of $G(s)$ (Fig. 38) used will depend on the characteristics of the velocity reference signal available for damping. In each case it is possible to carry out a study showing the optimum form of $G(s)$ for minimum position error considering the mission to be flown.

While each aided inertial system has some advantages, careful operational studies will usually uncover defects as well. In practice there is no generally acceptable substitute for an excellent basic inertial navigation system.

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THE FRANKLIN INSTITUTE

MINUTES OF THE STATED MEETING

October 15, 1958

The Stated Meeting of The Franklin Institute was held at 8:15 P.M. in Franklin Hall, following the Annual Medal Day Reception and Dinner, at which 402 members and guests were present.

Mr. Wynn Laurence LePage, President, called the meeting to order and announced that the minutes of the May Stated Meeting had been printed in the June issue of the JOURNAL OF THE FRANKLIN INSTITUTE. There being no corrections or additions, the minutes were approved as published.

President LePage spoke of the mutual interest and understanding which exists among men of science the world over, emphasizing that creative instinct when practiced leaves no room for distrust and conflict. He brought to the attention of the members and friends of The Institute the pamphlet on the "Awards of The Franklin Institute," noting that in it would be found names of scientists and engineers from many countries, and expressed the wish that in this great Hall, inspired by the memory of Benjamin Franklin, men of superior understanding in international affairs could meet in the same calm atmosphere that prevails when scientists from all over the world gather together to exchange ideas and theories concerning the origin, structure and exploration of our vast God-given Universe. In the *Institute News* he noted frequent mention, in words and pictures, of scientists of countries around the world who have visited here in friendly company with our own men of science. In paying tribute to the men to be honored, President LePage referred to the writings of Benjamin Franklin who said that a scientist should be able to set foot in any country in the world and call it his own.

The Sponsors then presented the recipients of awards, and the Presentation of the Medals was made by President LePage.

Following the Presentation, Mr. Donald Wills Douglas, Chairman of the Board of the Douglas Aircraft Company, Inc., of Santa Monica, California, recipient of the Franklin Medal, gave a very interesting talk on "Space, Science and Education." In his talk Mr. Douglas suggested that to meet the Russian challenge we must go further than we have in establishing incentive for real inventors and innovators.

The meeting was adjourned at 10:10 P.M.

WILLIAM F. JACKSON, JR.
Secretary

NOTE: The entire proceedings of Medal Day will be published in the December issue of the JOURNAL.

MEDAL DAY PROGRAM

Reception for Medalists ..... THE HOSTESS COMMITTEE
Invocation ..... THE REVEREND JOHN C. KNEWSTUB
Dinner ..... MEDALISTS, OFFICIALS AND GUESTS
Toast to Benjamin Franklin ..... WYNN LAURENCE LEPAGE, *President*
Introduction of Former Medalists ..... JOSEPH GRAY JACKSON, *Chairman, Committee on Science and the Arts*
Stated Meeting of the Institute ..... THE PRESIDENT PRESIDING
Coming Events at the Institute ..... JOHN S. BURLEW, *Executive Vice President*
Presentation of Medalists ..... COMMITTEE ON SCIENCE AND THE ARTS
Awarding of Medals ..... THE PRESIDENT
"Space, Science, and Education" ..... DONALD WILLS DOUGLAS

1958 MEDALS

Cresson Medals (1848)

JOSEPH C. PATRICK

Consultant
Thiokol Chemical Corporation
Trenton, New Jersey

Synthetic Rubber and Rocket Fuel

S. TIMOSHENKO

Professor Emeritus
Stanford University
Stanford, California

Contributions to Theory of Elasticity and Elastic Stability

Certificate of Merit (1882)

HAROLD LYONS

Head, Atomic Physics Department
Hughes Aircraft Company
Culver City, California

Atomic Clock

Longstreth Medals (1890)

JAMES BAILEY

Consultant
Plax Corporation
Hartford, Connecticut

Unbreakable Plastic Bottle

GEORGE S. CRAMPTON

Ophthalmologist
Philadelphia, Pennsylvania

Borescope

PRICE C. MCLEMORE

Planter, Agricultural Consultant and Inventor
Montgomery, Alabama

Flame Weeding in Cotton Fields

Potts Medals (1906)

W. NELSON GOODWIN, JR.

Consultant
Weston Electrical Instrument Corp.
Newark, New Jersey

Photoelectric Exposure Meter and Electrical Instruments

EMANUEL ROSENBERG

Electrical Engineer
Bogotá, Colombia, South America

Cross-field Electrical Generator

Henderson Medal (1924)

ARTHUR H. MOREY

Manager, Locomotive Advance Engineering
Locomotive and Car Equipment Department
General Electric Company
Erie, Pennsylvania

4500 h. p. Gas Turbine-Electric Locomotive

Wetherill Medals (1925)

H. A. H. BOOT

Senior Principal Scientific Officer
Services Electronics Research Laboratory
Baldock, England

Pulsed Microwave Magnetron

J. T. RANDALL

Wheatstone Professor of Physics
University of London King's College
London, England

Pulsed Microwave Magnetron

J. SAYERS

Professor, Electron Physics Department
The University of Birmingham
Birmingham, England

Pulsed Microwave Magnetron

Clark Medal (1926)

FREDERIC O. HESS

President
Selas Corporation of America
Dresher, Pennsylvania

New Processes for Use of Gas Industrially

Brown Medal (1938)

CHARLES M. SPOFFORD

Professor Emeritus
Massachusetts Institute of Technology
Cambridge, Massachusetts

Outstanding Civil Engineering Contributions

Clamer Medal (1943)

JULIAN MILES AVERY

Metallurgican and Chemical Consultant
Ethyl Corporation
New York, New York

High Top-pressure Operation of Blast Furnaces

Ballantine Medal (1946)

HARALD T. FRIIS

Consulting Engineer
Rumson, New Jersey

Contributions to Radio Communications

The Franklin Medal (1914)

DONALD WILLS DOUGLAS

Chairman of the Board
Douglas Aircraft Company, Inc.
Santa Monica, California

Creative Engineering in the Field of Aeronautical Design

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Ultrasonic Fatness Tests on Live Pigs.—An ultrasonic wave machine is being used by the English film of British Oil and Cake Mills to measure the thickness of fat on the back of live pigs. This new British wave method is claimed to be accurate, humane, safe and simple.

A high-frequency sound wave, well above audible frequencies, is sent through the back fat of the pig, and on reaching the boundary between the fat and lean tissues it is reflected back to the surface. The time taken for the sound to go through the fat and back again is a measure of the fat thickness. The same principle is used for measuring the depth of fish shoals by echo-sounding at sea. The instrument used is an ultrasonic flow detector designed to measure the depth of flaws or impurities in metal castings.

The method of testing is as follows: A probe which is placed on the pig's skin contains two barium titanate crystals. One sends out sound waves and the other receives the echo. The transmitting crystal is made to vibrate by the application of high-frequency electrical pulses of 2½ mc/s. These vibrations produce the requisite sound waves. The receiving crystal picks up the echo and converts the sound back to electrical pulses. The crystals are connected by long leads to the main instrument where the transmitted and reflected pulses are amplified and displayed as narrow peaks on a cathode ray tube. The distance between peaks is proportional to the fat thickness.

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THE FRANKLIN INSTITUTE TESTING MACHINE—1832

BY
THOMAS COULSON[1]

A small area in the Museum, called the Hall of Honor, is reserved for articles and apparatus intimately associated with distinguished scientists, engineers, and inventors. One of the exhibits displayed here has to do with no individual, but is closely related to The Franklin Institute, its members and the spirit which has animated their work through the long years of its existence. This is the Testing Machine, the first machine for testing the strength of materials in this country.

In the early days of the Institute, the members displayed an unusual interest in research, unusual because it departed from the conventional procedure. Scientific research workers were prone to keep very much to themselves, working in seclusion in their cabinets, and concentrating on the laws of nature and the nature of things, without reference to the practical application of the knowledge they accumulated. This detachment from the world of practice is demonstrated in the growth of the steam engine which grew in the hands of practical engineers whose scientific knowledge was, to say the least, restricted.

Once The Franklin Institute was founded a movement was started, consciously or unconsciously, to break down this attitude of "the ivory tower" and to resort to concerted research of a nature that would have immediate practical value. Only those problems that were agitating industry at the time were acceptable to the members. The shipping industry was flourishing in Philadelphia at the time, so that it became natural to undertake a study of dry docks. This piece of research was accomplished with such gratifying results that the members were encouraged to approach, on a more ambitious scale, a fundamental feature of the nation's growing industries, its prime sources of power. One committee was appointed to conduct researches into the design and operation of water wheels and, shortly after, another was appointed to investigate the causes of steam boiler explosions. It was this latter which produced the Testing Machine.

With Oliver Evans introducing the use of high pressure steam, the numbers of steam boiler explosions became a threat to the further progress of steam applications. Travellers were hesitant to embark on Mississippi steamboats because of the peril, largely increased by the captains' irresistible passion for racing their boats. Railroad companies tried to re-assure their passengers by loading the leading car with bales of hay, and one southern company, whose directors had more delicate

[1] Director of Museum Research.

sensibilities, gave this place to a negro orchestra, whose function, apart from providing music, was to absorb the initial violence of the shock when the boiler let go.

The members of the Institute were resolved upon finding the cause of these explosions, and the committee they appointed went a step farther in trying to establish a standard for boiler construction instead of leaving this to the caprices of individual boilermakers. Opinions varied very considerably upon such fundamental elements as the material to be employed. It was to establish a minimum standard of safety in the quality and thickness of the metals (iron and copper) used in boiler construction that the members of the committee designed the Testing Machine of 1832.

Apart from accumulating a vast amount of information relating to steam boiler explosions, the work of this committee was noteworthy for two other reasons, both springing from the recognition of the work by the Federal government. The practical work had been in progress only a short time when its value was realized, and the sum of $1500 was appropriated to promote it. This, as far as can be ascertained, was the first time the Federal government appropriated funds for scientific research other than in the field of medicine.

The second notable feature was that, as the outcome of their labors, the committee was able to furnish the Senate with the draft of a Bill, which was enacted and became the Federal law governing the inspection of steam boilers and engines on steamships.

This did not by any means exhaust the interest of the membership in the subject, nor did it terminate the use of the Testing Machine. The apparatus was held in such high esteem that, after it had served its original purpose, it was set up in the old building on Seventh Street, where it was made accessible to anyone who wished to test the strength of iron and copper which they contemplated employing.

It is fitting that this unique apparatus should be preserved in the Hall of Honor, which it shares with relics from the labors of distinguished men who have advanced our knowledge in various fields of science and technology.

BARTOL RESEARCH FOUNDATION
W. F. G. SWANN, DIRECTOR

PROPOSED SYSTEMATICS FOR E2 TRANSITION PROBABILITIES OF EVEN-EVEN NUCLEI *

BY
D. M. VAN PATTER

It is well known that collective effects have been observed concerning the lifetime of the first level of even-even nuclei (1). In addition to a strong enhancement of the E2 transition probability, or reduction of the lifetime, for strongly deformed nuclei in the rotational region, the enhancement also diminishes as the neutron shell $N = 126$ is approached (1). It is also known that the value of E_1, the energy of the first excited state, increases as the $N = 126$ shell is approached (2). Similar effects have also been observed from studies of Coulomb excitation cross sections (3) in the regions where the nuclei exhibit vibrational spectra. For nuclei in the vibrational region approximately defined by $A = 60$–150, the ratio E_2/E_1 of the energies of the first two excited states shows a fairly smooth variation with E_1 (4). Since such collective effects are known to be present, it seems possible that a similar systematic trend with E_1 might be found for the E2 transition probability for the first level of even-even nuclei.

A compilation has been made of all available data concerning lifetime determinations and Coulomb excitation cross sections, which includes measurements for some 90 nuclei ranging from C^{12} to U^{238}. The results to be given are preliminary in nature, since a complete analysis of all the data has not been finished. However, in view of the current interest regarding systematic trends (5,6), it was felt that a preliminary note should be submitted at this time.

A preliminary analysis of the data concerning E2 transition probability has yielded the following results. For nuclei with $A = 46$–132, the reduced E2 transition probability for excitation of the first $2+$ level can be well represented by the following expression:

$$\frac{B(\mathrm{E2}; 2\rightarrow 0)_{\mathrm{exc}}}{Z^2} = 1.1 \times 10^{-4} E_1^{-1.4\pm 0.2} \text{ (in units of } e^2 \times 10^{-48} \text{ cm}^4\text{)}. \quad (1)$$

Of the some forty nuclei in this mass region for which measurements have been made, over 80 per cent have values of $B(\mathrm{E2})/Z^2$ which lie within ± 25 per cent of the value given by Eq. 1. The values for the nuclei Se^{82} and $Zr^{92,94}$ seem to be significantly lower, evidently because they have neutron numbers differing by only 2 from the $N = 50$ shell.

* This research was supported by the U. S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

However, it should be noted that shell effects for $B(E2)$ have been largely absorbed into the E_1 variation, and that the values of $B(E2)/Z^2$ for seven stable isotopes of tin ($Z = 50$) are well represented by this expression.

It should be pointed out that this apparent correlation between $B(E2)/Z^2$ and $E_1^{-1.4}$ is not unique. For the same nuclei, similar correlations can be exhibited for the favored factor F, and for $FA^{5/3}/Z^2$, where:

$$F = B(E2)/B(E2)_{sp} \quad B(E2)_{sp} \simeq 3 \times 10^{-5} A^{4/3} \times 10^{-48} \text{ cm}^4.$$

This can be explained by noting that $A^{5/3}$ and Z^2 are approximately equal for these stable even-even nuclei, and hence it is difficult, if not impossible, to distinguish between these functions using the present experimental data. Coleman (6) has recently independently noticed that F varies approximately as $E_1^{-2.0}$ for nuclei in the region $A = 46$–132. This variation of $E_1^{-2.0}$ was based on the results of Temmer and Heydenburg (3), while the variation of $E_1^{-1.4}$ given in (1) is primarily based on more recent results, particularly those of Stelson and McGowan (5). These latter also noticed that for nuclei in the region $A = 94$–130, the values of $B(E2)$ appeared to be correlated with E_1 (5). However, when the values of $B(E2)$ for nuclei in the lower mass region $A = 46$–80 are included, a multiplying factor of approximately $1/Z^2$ or $1/A^{4/3}$ is necessary for correlating the data with E_1.

A survey of the values of $B(E2)$ for nuclei with $A = 154$–206 has also been made, and a similar trend of $B(E2; 2 \rightarrow 0)/Z^2$ varying approximately as $E_1^{-1.4}$ has been noted, although this is not as clear-cut. Again, the nuclei Hg^{204} and Pb^{206}, with neutron numbers differing only by 2 from the $N = 126$ shell, have significantly lower values of $B(E2)/Z^2$. Another interesting feature has been noticed for the isotopes of W, Os and Pt. The value of the reduced transition probability, $B(E2; 2' \rightarrow 2)/Z^2$ for the E2 transition between the second $2+$ level and the first level follow the same general trend of approximately $E_1^{-1.4}$ as the values of $B(E2; 2 \rightarrow 0)/Z^2$. On the other hand, the values of $B(E2; 2' \rightarrow 0)/Z^2$ are significantly lower and show no particular trend with E_1.

At the present time there does not seem to be any obvious theoretical explanation of these apparent systematics for the E2 transition probabilities for even-even nuclei. However, it does seem possible that the E2 transition probabilities, or the lifetimes, of the first $2+$ levels of even-even nuclei in the mass region $A = 46$ to 132 can be predicted within ± 25 per cent with few exceptions. The author is indebted to several people, particularly Dr. P. H. Stelson and Dr. B. Elbek for the communication of their results prior to publication. In addition, many stimulating discussions, particularly with regard to the predictions of the collective model, were provided by Prof. B. James Raz.

REFERENCES

(1) For example, see A. W. SUNYAR, *Phys. Rev.*, Vol. 98, p. 653 (1955).
(2) G. SCHARFF-GOLDHABER, *Phys. Rev.*, Vol. 90, p. 587 (1953).
(3) G. M. TEMMER AND N. P. HEYDENBURG, *Phys. Rev.*, Vol. 104, p. 567 (1956).
(4) G. SCHARFF-GOLDHABER AND J. WESENER, *Phys. Rev.*, Vol. 98, p. 212 (1955).
(5) P. H. STELSON AND F. K. MCGOWAN, *Phys. Rev.*, Vol. 110, p. 489 (1958).
(6) C. F. COLEMAN, *Nuclear Physics*, Vol. 7, p. 488 (1958).

BOOK REVIEWS

THE SCIENCE OF HIGH EXPLOSIVES, by Melvin A. Cook. ACS Monograph 139. 440 pages, illustrations, 7 × 10 in. New York, Reinhold Publishing Corp., 1958. Price, $22.50.

Until comparatively recently the literature on high explosives has consisted of a considerable volume of empirical work on the preparation, properties and uses of explosives and a small number of highly mathematical papers attempting to explain the phenomenon of detonation. The gulf between the empirical technologist and the scientist was wide. Few of the people handling explosives were willing and able to understand the mathematics of the theorists, and the scientists were not able to fit data to their equations that would convert their findings to practical applications.

During and since World War II, military high explosives applications have become enormously more complicated, requiring not only greater power but also extreme precision. The empirical approach is not well suited to the solution of such problems. Accordingly, more effort and support have been directed toward the theoretical approaches. A group of laboratories has grown up with highly trained specialists and elaborate, precise and expensive instrumentation. These laboratories have derived their support mainly from the Armed Services and the AEC and published their work largely under security classification. Some of these laboratories have now opened their doors to non-military customers. Some of the work has reached the open literature, and more could be published by overcoming the inertia of authors.

The publication in book form and with consistent nomenclature of results of modern research in the explosives field is timely. It makes readily available to the developers and users of non-military explosives a wealth of material that should lead to better understanding of empirical facts and should afford clues for further development.

Following a detailed description of the instrumentation available in the author's laboratory and illustrations of the type of data obtained, Cook examines the classical hydrodynamic theory of detonation. Although boundary conditions, that is, initial state and Chapman-Jouquet plane, are pretty well settled once the equation of state is known, there is still uncertainty about what goes on between. Cook attacks the Zeldovich–von Neumann–Doering spike theory and offers his heat pulse theory as an alternate, together with experiments supporting the heat pulse theory. He next introduces his concept of the detonation head. This is the region bounded on the front by the detonation shock wave, in the rear by the Langweiler rarefaction wave and on the sides by lateral rarefaction waves. In the case of ideal explosives in steady-state detonation, the lateral rarefaction waves intersect at a point on the axis about 3.5 diameters behind the front, so the terminal rarefaction wave is no longer effective. Within the detonation head, density at one third greater than that of the unreacted explosive and also temperature and pressure are roughly constant. The detonation head behaves much as a solid traveling at detonation velocity and produces on a target much the same effects as would be produced by a solid body traveling at that velocity.

Condensed phase explosive particles, droplets, etc. within the detonation head burn inward from their surface by parallel layers, following the same burning law as propellants, but much more rapidly because the pressure in that region is very high. If the C-J plane (reaction complete) lies within the detonation head, the detonation is called "ideal." If, on the other hand, reaction is not complete before rarefaction stops it, the detonation is called "non-ideal." Chemical reactions not completed within the detonation head can proceed later more slowly. This portion of Cook's theory was developed independently a quarter of a century earlier in Germany.

Gases produced in the detonation reaction should attain thermochemical equilibrium in the detonation head. Hence the equilibria and the heat capacities of the individual

species must be considered in calculating detonation pressures and temperatures. A considerable body of thermochemical data is presented in the text and particularly in Appendix II. The gaseous products will continue to react during their expansion after detonation, but will depart from equilibrium before ambient pressures are reached. This accounts for the presence of small quantities of peculiar chemicals in the fumes resulting from blasting.

As in any other combustion process, ions and electrons are found in detonation reaction zones. These may be used to produce electromagnetic effects in the surroundings. They offer a means of accomplishing Cook's heat pulse.

Having explained steady state detonation as a phenomenon propagated by a supersonic shock wave, we turn to the question of what starts a detonation. It has been pretty well established by the work of many investigators that the phenomenon always starts from a fire, or deflagration. Deflagration in turn may be caused by thermal decomposition at a high temperature, by a hot spot produced in adiabatic compression of trapped gases during an impact, or by other mechanisms. In general deflagration starts at "hot spots" rather than uniformly throughout the explosive. The difficulty of reproducibly producing hot spots is the reason for the scatter of results with impact sensitivity test machines and for lack of agreement between test procedures for ranking explosives according to sensitivity.

If an explosive will proceed from deflagration to detonation within a short time and small dimensions, it is known as a primary explosive and its detonation shock can be used to initiate less sensitive explosives. Despite a great deal of work in different laboratories, the actual mechanism of setting up the detonation shock wave remains imperfectly explained. Cook's work with condensed explosives on this subject is restricted to experiments where a significant shock wave accompanied the deflagration, thus could at best pertain only to the final stage of the transit.

Metallurgical properties of solids under explosive attack are discussed, and related to the metallurgy of slugs formed in the firing of lined-cavity shaped charges. The shaped charge is described, with proper emphasis on cavity liners, and a comprehensive treatment given of shaped charge design parameters. The mechanism of the shaped charge phenomenon is explained in terms of the Cook detonation head.

In blasting, use is commonly made of the lateral component of the expansion of the detonation products. Good field practice recognizes that for some applications the prolonged application of force from non-ideal detonation may be preferred to the greater brisance of ideal detonation, that it is important in any event to attain high loading densities in bore holes by tamping or stemming. These and other empirical facts can be predicted from theoretical bases.

Shock waves leaving a detonating explosive will be transmitted to the surroundings, which will be gaseous (air), liquid (water) and/or solid (earth, rock, metal). Discussion is presented of what effects these shock waves can produce in these media. Various mechanisms of rupture of rock are noted and the relation shown between type of blasting and type of fracture accomplished.

The text concludes with a discussion of damage potential. Damage can occur as a result of fragments thrown, or from blast waves transmitted through the air or through the ground. At a distance, air shock appears to be more important than ground shock. There is some uncertainty as to whether the property of causing damage resides in pressure, surface energy density or impulse. This is a matter of some importance on account of its bearing on the problem of the proper separation of structures containing explosives from each other and from other structures, railroads and highways. If pressure is overriding, the distances should be proportional to the $\frac{1}{3}$ power of the weight of explosive. The American Table of Distances is based on this choice. If surface energy is the important factor, a $\frac{1}{2}$ power law should apply. If impulse is the villain we should use a $\frac{2}{3}$ power law. Records of damage from accidental shots are plotted, with enough scatter to justify any one of the three choices (Cook advocates the $\frac{1}{3}$ power law). Glass breakage data and earth craters support the pressure mechanism. Plaster damage data support the surface energy mechanism. There is also evidence, not quoted by Cook, to support the impulse mechanism.

Not treated except by inference is the

problem of proper design structures containing explosives. By analogy to the problem of blasting we infer that low loading densities inside barricades should be advantageous.

The amazing scope of this work is noted, as is also the thread of unity running through the theoretical development out to the practical applications. Whether or not one accepts all of Cook's theory, it cannot be denied that this system affords working hypotheses that predict a wide variety of known effects and will probably lead to the discovery of new phenomena.

Noted also is that the chief sources from which this book is derived are the work of his own laboratory (one fifth of all references) for theory and the duPont Blasters Handbook for practice. Portions of mathematics and tabulated data that were spot-checked showed typographical errors.

Since there are other authorities working in the field, who must disagree with Cook as much as Cook disagrees with them, it may be expected that more good books on explosives may appear to add to our libraries.

A. M. BALL
Hercules Powder Co.

AUTOMATIC PROCESS CONTROL, by Donald P. Eckman. 368 pages, diagrams 6 × 9 in. New York, John Wiley & Sons, Inc., 1958. Price, $9.00.

This is a basic book for the student engineer in automatic controls with a working knowledge of higher mathematics. Considering only the analysis of feedback control systems, it is merely another good text. However, it has at least two unusual features which raise it above the ordinary. First, it applies control theory intensely to process controls by way of many examples and problems (with answers). Second, the author has made each chapter stand alone, permitting flexibility in the sequence and emphasis of a study program. For example, if the emphasis is to be on a mathematical approach, he suggests the following order: Chapter 1, appendix, and Chapters 4, 9, 10, 5, 7 and 8.

The author is no newcomer to the field. Wiley has previously published two of his books, "Principles of Industrial Process Control" and "Industrial Instrumentation." Some thirty of his technical papers and patents have been published elsewhere. He holds an MS in Mechanical Engineering, took his Doctorate in Automatic Control and worked in the field for 7 years. He also has more than 10 years of experience in teaching instrumentation and automatic control.

The text begins with an interesting historical background including a set of references. It seems that the first intelligent application of automatic controls was the flyball governor on Watt's steam engine in 1775. The first mathematical analysis was of the flyball governor by James Clerk Maxwell in 1868.

The first two chapters present the usual case for automatic control extending it into basic philosophy of control. Chapter 3 treats basic types of controllers and their characteristics. This is followed by a chapter on closed loops and the evaluation of static and dynamic error coefficients. In it the author promotes an unconventional method, namely, the Ziegler-Nichols method of calculating behavior from experimental data.

Chapter 5 covers the subject of measuring (feedback) elements and the following two are devoted to compensating and final control components. Here, the author recognizes reality in the problem of making a system fail-safe. The "Process and Instrumentation" chapter which follows is a section for discussion of general system behavior under certain given conditions.

Chapter 9 begins the more abstract portion of the text. Sinusoidal analysis is developed from classical differential equations and operational calculus. Bode, polar, and Nichols charts are employed to illustrate and implement the solution of typical frequency response problems. The final chapter is a conventional treatise on stability analysis including the old warhorses, Laplace, Routh-Hurwitz, Bode and Nyquist.

The appendix is a bonus for the student as well as the practicing engineer. It contains frequently used normalized transient and frequency response curves, Laplace transforms, analog computer techniques, and a series of laboratory experiments amplifying the theory.

In summary, the book is an excellent one for implementing a course in automatic control and applying modern basic theory to process controls. The over-all quality, technical and physical, conforms to the high standards of the publishers.

CHARLES A. BELSTERLING
The Franklin Institute Laboratories

SOLID STATE PHYSICS. SUPPLEMENT 1, NUCLEAR QUADRUPOLE RESONANCE SPECTROSCOPY, by T. P. Das and E. L. Hahn. 223 pages, illustrations, 6 × 9 in. New York, Academic Press, Inc., 1958. Price, $7.00.

Considerable material exists in the literature describing an analysis of compounds and crystals based on magnetic interactions. It is only recently that nuclear electric quadrupole interactions have begun to be utilized as an analytical tool. The fundamental theory of nuclear quadrupole interactions has been presented by M. H. Cohen and F. Reif in Volume V of the Solid State Physics Series. However in Cohen and Reif's work, the nuclear quadrupole interactions were treated as a small perturbation to magnetic interactions. The present monograph emphasizes the case where the electric quadrupole interactions are strong (strong enough to give rise to r.f. resonance frequencies). It is divided in three parts.

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Part I (about 80 pages) presents the theory of pure quadrupole interactions, namely: frequencies and intensities of their spectra, their splitting and broadening, the effect of internal motions in molecular solids on resonance and the theory of transient experiments in Quadrupole Resonance. Part II (about 15 pages) describes the required instrumentation and circuitry.

Part III, which justifies the object of the monograph, describes applications of electric quadrupole resonance, of interest in Solid State. This is a well written and most useful presentation of a new efficient tool to investigate the properties of solids. Let us mention some of the applications: Information on the crystalline unit cell, electron distribution in molecules, intermolecular binding in solids, role of impurities etc. Nineteen tables increase the usefulness of this exposé. Those who wish to use this new investigative tool will find the monograph a thorough introduction to its use.

Since this is a specialized technique, the editors of the Solid State Physics Series are to be praised for publishing it in a separate supplement. Judging from the excellence of this first supplement, one may expect a great demand for similar monographs.

HENRI AMAR
The Franklin Institute Laboratories

PHYSICS OF NUCLEAR FISSION, edited by E. Bretscher and D. J. Hughes. 182 pages, diagrams, plates, 5½ × 8½ in. New York and London, Pergamon Press Inc., 1958. Price $9.00.

This book is an English translation, by Dr. J. E. S. Bradley, of a special supplement to the Soviet journal *Atomnaya Energiya* containing papers presented at the conference on fission physics held at the Atomic Energy Institute of the U.S.S.R. Academy of Sciences in January 1956. There are twelve chapters, written by thirteen Soviet physicists, which deal with basic aspects of the nuclear fission process. While some of these aspects are directly related to the application of nuclear fission in chain-reacting systems, the applications themselves are not discussed and the book contains no reactor physics. Nearly all the papers are in the form of review articles. The topics reviewed are: nuclear fission theory, fast neutron fission cross sections, charge and mass distributions of fission products, anisotropy in fission processes (including angular distribution of the fission fragments), properties of neutrons emitted in fission (including time of emission, energy spectrum, angular distribution, and number emitted per act of fission), ternary and quaternary fission, high energy proton fission and spallation, spontaneous fission, and photofission.

A great deal of material is covered in this 182-page book and the treatment is extremely concise in most chapters. In general, the articles are well written (apart from overconciseness) and the translation appears to be uniformly good. Particularly valuable are the very complete set of references to the literature provided at the end of each chapter and also the large number of figures and tables collecting results of many workers. Significant pre-war contributions to fission physics were made by the Soviet physicists Frenkel, Flerov and Petrzhak; one is not surprised to find that the Russians have continued to work very actively on all phases of the subject. The book unfortunately lacks an index and contains an unusual number of misprints.

R. D. PRESENT
The University of Tennessee

NUCLEAR SCATTERING, by K. B. Mather and P. Swan. 469 pages, diagrams, 5½ × 8½ in. New York, Cambridge University Press, 1958. Price, $14.50.

This very interesting exposition of the many facets involved in the scattering of nuclear particles is not for the beginner. The text is lucid, but a considerable knowledge of the various particles and their properties and of the mathematics of quantum mechanics is assumed.

Nuclear structure and nuclear forces are reviewed quickly in the first chapter. The following four chapters discuss briefly the various experimental methods that have been used. The emphasis is on general principles, and the few graphs and tables giving results seem almost incidental. Mathematical equations are given for many of the results or for the factors involved, but very few of these are derived. The remaining seven chapters discuss the various types of scattering: neutron-proton, proton-proton, neutrons and protons by very light nuclei, non-central forces, high energy scattering by light nuclei, resonance scattering. The form of the wave equation applicable to each case is given, together with other mathematical restrictions. The results are stated mathematically, with a minimum of derivation. Frequently, the theoretical and experimental results are compared graphically.

This book will provide a valuable summary of the field for any one with sufficient background to understand it. Even those not able to understand all of the details should find many of the discussions to be informative. There are not sufficient details to be of value to anyone working in the field.

NEWTON MCCREADY
Philadelphia Quartz Co.

NONLINEAR CONTROL SYSTEMS, by R. L. Cosgriff. 328 pages, diagrams, 7 × 9 in. New York, McGraw-Hill Book Co. Inc., 1958. Price, $9.00.

Professor Cosgriff apparently visualizes the evolution of a new kind of specialist which he calls the "control engineer." This man may very well exist and would probably be distinguished by an accurate knowledge of transducers, servomechanisms, switching logic, and data processing. It is this specialist's competence in servomechanisms which Prof. Cosgriff seeks to reinforce. Pointing out that there is very little material which has been prepared in text form covering nonlinear systems, he undertakes to investigate the various kinds of nonlinearity with which the control engineer is confronted. The scope, therefore, includes more than the limitations of electronic amplifier circuits in servo systems, but goes further to treat backlash, relay controls, nonlinear friction and numerous forms of control devices requiring compensation. In this sense the book is designed more precisely for the control engineer than many previous texts, primarily devoted to the theory of servomechanisms, have been.

It is interesting to analyze the materials presented in this volume. For example, the author has been quick to recognize the growing importance of logic circuits as control elements. Also the application of statistical methods, although rather brief in treatment, as is the part on logic circuits, greatly broadens the engineer's approach to control problems. This is particularly true since most practical control systems contain some form of noise whose presence will be dealt with through the use of autocorrelation techniques.

Only a small portion of the book is devoted at the beginning to review of linear systems techniques. This information is immediately applied to the solution of nonlinear problems by taking the solution in parts. This approach is extended to relay control systems which can be linearized, and to cases of nonlinear friction. Nonlinear feedback methods for improving system performance are treated through the application of phase plane analysis. This analysis is limited to second-order differential equations, and the more general concepts of phase space are avoided.

The effectiveness of Prof. Cosgriff's presentation is assured by his apparent conviction regarding the needs of the control engineer. The formal developments in the text are essentially theoretical and the working of examples is relegated to the problem sections. However, in this way he has been able to supply the strong fundamental background required in the wide variety of control engineering and at the same time keep the book conveniently small. It would not appear possible, however, to gain a working knowledge of the methods presented without solving most of the problems.

One might predict that other books will

begin to appear covering the ground which Prof. Cosgriff's text suggests. That is, there is a need for instruction in the broad aspects of control system design embracing both analog and digital methods as well as systems which, as is so often the case now, include a combination of the two. The present volume comes close to this goal and will be welcomed by those engaged in the conception and original development of control systems.

C. W. HARGENS
The Franklin Institute Laboratories

ELECTRONIC SEMICONDUCTORS, by Eberhard Spenke, translated from the German by D. Jenny *et al.* 402 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co. Inc., 1958. Price, $11.00.

This is a translation of the second German edition of *Elektronische Halbleiter*. The book has two main divisions: (1) "The Conduction Mechanism of Electronic Semiconductors and the Physics of Rectifiers and Transistors" and (2) "Fundamentals of Semiconductor Physics." These sections are radically different in style and method of treatment, and are even, as is frankly admitted in the preface, directed toward different groups of readers. This leads one to question the wisdom of binding them together in one volume.

The first section, comprising Chapters I to V, is directed to the "young technically trained person who has a special interest in the specific problem covered" that is, rectifiers or transistors. Much more care has been devoted to the second section, Chapters VI to X, which is intended for more advanced students. The treatment of the band model in Chapter VII is especially good. The illustrations throughout are well chosen and of excellent quality. Unfortunately for this reviewer the over-all effect was spoiled by a somewhat contrived effort to explain the zero conductivity of a full band as due to a cancellation by the current of negative effective mass electrons at the top of the band of the current due to the positive effective mass electrons at the bottom of the band (see, for example, Chapter I, p. 11). It is questionable whether this can be justified even for pedagogic purposes. The accepted explanation is not given until Chapter VII, Section 10.

F. J. DONAHOE
The Franklin Institute Laboratories

PUBLICATIONS RECEIVED

NUMERICAL MATHEMATICAL ANALYSIS, by James B. Scarborough. Fourth edition, revised, 576 pages, diagrams, 6 × 9 in. Baltimore, The Johns Hopkins Press, 1958. Price, $6.00.

INTRODUCTION TO METEOROLOGY, by Sverre Petterssen. Second edition, 327 pages, illustrations, plates, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1958. Price, $6.75.

TOPOLOGICAL GROUPS, by Leon Pontrjagin. 299 pages, 6 × 9 in. Princeton, Princeton University Press, 1958. Fifth printing. Price, $2.75 (paper).

THE PRINCIPLES OF SCIENCE, by W. Stanley Jevons, with a new introduction by Ernest Nagel Unabridged and unaltered republication of the latest edition, 786 pages, 5½ × 8 in. New York, Dover Publications, Inc., 1958. Price, $2.98 (paper.)

AN INTRODUCTION TO THE STUDY OF EXPERIMENTAL MEDICINE, by Claude Bernard, with a new foreword by I. Bernard Cohen. Unabridged and unaltered republication of the first English translation. 226 pages, 5½ × 8 in. New York, Dover Publications, Inc., 1958. Price, $1.50 (paper).

SCIENCE THEORY AND MAN, by Erwin C. Schrödinger. Unabridged reproduction of the work first published in 1935 under the title "Science and the Human Temperament," 222 pages, 5½ × 8 in. New York, Dover Publications, Inc., 1958. Price, $1.35 (paper).

RESEARCH ON POWER FROM FUSION AND OTHER MAJOR ACTIVITIES IN THE ATOMIC ENERGY PROGRAMS, JANUARY–JUNE 1958. 410 pages, illustrations, 6 × 9¼ in. Washington, U. S. Atomic Energy Commission, 1958. Price, $1.25 (paper).

REVIEW OF COLLEGE PHYSICS, VOLUME I, MECHANICS, WAVE MOTION, SOUND, by Robert L. Weber. 146 pages, diagrams, 6 × 9 in. Englewood Cliffs (N. J.), Prentice-Hall, Inc., 1958. Price, $2.25 (paper).

BOOK NOTES

FLAVOR RESEARCH AND FOOD ACCEPTANCE, sponsored by Arthur D. Little, Inc. 391 pages, diagrams, 6 × 9 in. New York, Reinhold Publishing Corp., 1958. Price, $10.00.

Based on a series of four symposia sponsored by Arthur D. Little, Inc., during 1956–1957, this book is a survey of flavor and associated research. Subjects covered include sensory testing, consumer testing, psychology, physiology and chemistry. This is the first work to bring together the several approaches to flavor research, and as such it will be useful to all engaged in the food industry. The book is divided into five parts: Physiological and Psychological Aspects (42 pages); Laboratory Flavor Testing (79 pages); Consumer Product Testing (59 pages); Specific Applications of Product Testing in Industry (64 pages); and Physicochemical Research (120 pages). The authors of the 35 papers are recognized leaders in the field.

THE HELICOPTER, by Jacob Shapiro. 269 pages, diagrams, plates, 5 × 8 in. New York, The Macmillan Co., 1958. Price, $4.50.

Aimed at the lay reader, this book explains the history, flight principles, design requirements, uses and future prospects of the helicopter. The author, an engineer and designer, writes with ease, in language readily understandable by non-specialists. In the chapter on "Outlook," the pros and cons of jet-driven helicopters are discussed.

TROUBLE-FREE HYDRAULICS, by Ian McNeil. 124 pages, diagrams, 5¼ × 8¼ in. New York, The Ronald Press Company, 1958. Price, $6.50.

This is a practical handbook on maintenance and fault finding in oil hydraulic machinery. Five chapters cover: an Introduction; Pipework and Oil Tanks; Hydraulic Fluids and Their Treatment; Hydraulic Seals and Packings; and The Diagnosis and Cure of Faults. Thirty-four tables and twelve fault-finding charts add to the usefulness of the handbook.

PHOTOGRAPHIC SENSITIVITY, VOL. 2, edited by Shin Fujisawa. 198 pages, illustrations, 7 × 10 in. Tokyo, Maruzen Co., Ltd., 1958. Price, $4.50.

Volume 2 in this series contains the 17 papers presented at a symposium held in 1957 at the University of Tokyo, and also carries notes and abstracts of papers presented at the 1954 and 1955 symposia. The full-length papers are divided among three sections: Electronic Process in Silver Halide Crystals (5 papers); Lattice Defects and Photolysis of Silver Halides (7 papers); and Sensitization of Photographic Emulsion (4 papers). A final paper by the Editor discusses future problems for research in the field.

QUANTITATIVE ANALYSIS, by R. A. Day, Jr. and A. L. Underwood. 465 pages, diagrams, 6 × 9 in. Englewood Cliffs (N. J.), Prentice-Hall, Inc., 1958. Price, $6.95.

One of the Prentice-Hall Chemistry Series, this new text by two members of the Department of Chemistry of Emory University, is intended to provide "nonchemist as well as specialist, with a background in analytical chemistry, reasonably proportioned with regard to both the basic, traditional material and the more recent advances." The three parts of the text cover volumetric analysis, gravimetric analysis and instrumental analysis. Six appendices cover tables of equilibrium constants, a bibliography, a mathematical review, log tables, etc.

VINYL RESINS, by W. Mayo Smith. 282 pages, illustrations, diagrams, 5 × 7¼ in. New York, Reinhold Publishing Corp., 1958. Price, $5.75.

Every attempt has been made to cover all recent advances in the vinyl field in this small handbook. New polyvinyl chloride "pearls"

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and Delrin resin are included. The treatment of polymerization systems for vinyl chloride is developed in considerable detail. Items such as vinyl laminates, regids, foamed material and latex paints, expected to show unusual growth, are clearly detailed. Here is a practical, informative text for manufacturers, fabricators and market developers.

MAGNETIC TAPE RECORDING, by H. G. M. Spratt. 319 pages, illustrations, 5½ × 8½ in. New York, The Macmillan Company, 1958. Price, $8.50.

Written especially for engineers who must be able either to build their own machines or to modify existing types, this volume is designed to cover the basic principles of magnetic recording and to enumerate the characteristics of both the medium and the machines. Chapters deal in principles of Magnetism and Magnetic Recording; Tape Recording Machines; Tape Testing; Recording Standardization and many other related topics. Two appendices on B. S. I. specifications and abnormal climatic effects are included, as well as an index.

ELECTRIC MACHINERY, by Clifford C. Carr. 537 pages, illustrations, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1958. Price, $9.25.

The approach used in this book is to establish the purpose of the machine, which is the conversion of energy. This is followed by an analysis of the feasible application of basic electric phenomena fulfilling the desired energy conversion. The text also discusses topics common to all electric machines: types of windings, voltage relations, magnetic-field relations, energy losses, energy flow, efficiency, ratings and basic torque relations. The remainder of the book is devoted to the application of these basic principles and a detailed analysis of the performance of the different types of machines.

THE REPRODUCTION OF COLOUR, by R. W. G. Hunt. 208 pages, plates, diagrams, 6½ × 8½ in. New York, The Macmillan Co., 1958. Price, $12.75.

The first half of this text covers the fundamental principles of photographic color reproduction, including television and printing. Presented without mathematics, the book's object is to present problems of a visual nature to those engaged in producing color pictures. The second half provides a more rigorous and detailed discussion of general color reproduction problems and also particular fields of application.

ELECTRONIC CIRCUITS, by E. J. Angelo, Jr. 450 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co. Inc., 1958. Price, $9.00.

This volume presents in considerable detail, the general techniques of developing piecewise-linear and incremental linear equivalent circuits, or network models, for physical devices. Modern circuit theory, augmented by certain new concepts which are directly related to the electronic devices, is used to develop a systematic theory for electronic circuits. The text provided some new approaches to many of the important topics in the study of electronic circuits.

INFORMATION AND COMMUNICATION PRACTICE IN INDUSTRY, edited by T. E. R. Singer. 304 pages, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1958. Price, $8.75.

In recent years, information-services groups have been playing a more important role in industrial communications. This handbook, prepared by a group of experts, sets forth the most efficient methods of dealing with the communications problem with particular reference to technical material. Chapters cover operations research, technical classification, research files, patent files, presentation of technical papers, punched-card methods, technical writing and editing, indexing and abstracting.

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The Editors wish to call attention to the series of mimeographed *Nuclear Technology Briefs*, prepared for and issued by the U. S. Atomic Energy Commission. Obtainable from the Division of Information Services, U. S. Atomic Energy Commission, Washington 25, D. C., the briefs provide American industry and science with authoritative (but by no means comprehensive) reports on important technological work as it emerges from research and development programs.

CURRENT TOPICS

Flash Heating Technique Initiates Thermal Reactions.—An important new technique for basic research on high temperature reactions was described at the American Chemical Society meeting in Chicago, by Drs. L. S. Nelson and J. L. Lundberg, Bell Telephone Laboratories. In this technique, a high intensity, high speed flash lamp is used to induce almost instantaneous heating of finely divided black body particles or filamentary materials immersed in transparent materials of various types.

Since only a few milliseconds elapse between initiation and quenching of the thermal reactions, the reaction intermediates do not undergo the usual degradation and condensation reactions which normally occur under longer exposure at high temperature conditions. Thus, information on the true reaction products should be forthcoming from studies using this new technique.

The work grew out of attempts to study photochemical reactions in polymers at higher and higher light intensities. The heating effect of the flash was soon found to exceed greatly the photochemical effect when insulated black bodies were present. This resulted from the high intensities used, and the use of a greater fraction of the lamp output by black body absorption as compared to photochemical absorption.

According to the authors, the new technique is applicable to the study of thermal reactions in solids, liquids, and gases. Basically, it utilizes black body heating of finely-divided or filamentary material to decompose either the black body itself or the surrounding matrix. For instance, fine black tungsten fibers can be suspended in mineral oil sealed in a tube. When this is placed within a helical capacitor flash lamp, and flashed at perhaps 1150 microfarads at 4000 volts, large deposits of carbon form from the pyrolyzed mineral oil. (If mineral oil without suspended black bodies is flashed, no discernible reaction occurs.)

Explosions have been initiated instantaneously throughout a sizable volume of combustible gas mixtures by this technique. Studies similar to shock tube initiation appear feasible, with the added advantage of a larger instantaneous number of unstable intermediate molecules in the reacting system.

If black body particles such as carbon or dust are immersed in a plastic such as polyethylene and flashed, the area immediately around the particle is pyrolyzed, giving off gases. These gases form bubbles in the viscous polymeric material. This flash technique could be used as a rapid test method to determine the amount of such impurities present in an apparently pure plastic material, and is now being studied.

In the cases cited above, the black body acts as an inert heat absorber, while the matrix reacts. The black body itself can enter into the reaction if desired, as well. For instance, if powdered coal is placed in an evacuated tube and flashed, it decomposes in an unusual fashion. Combustion products obtained in experiments of this type are all short chain hydrocarbons, characteristic of very intense heating of organic compounds.

Powdered metals can also be flashed in vacuum or in gas, and deposited on the walls of the tube in mirror surfaces. The authors feel that if certain powdered metals are flashed in the presence of hydrocarbon gases, metal alkyl compounds might result, along with a rich yield of free radicals.

The temperatures reached by the black bodies vary, depending on the intensity and duration of the flash, the size of the black body, and the insulating qualities of the matrix. Also, the presence or absence of phase changes during a flash has a marked effect on the ultimate temperature reached during an experiment. According to Lundberg and Nelson, temperatures at black body particles in low pressure gases may approach 6000 to 9000°F.

If the matrix undergoes a phase change from liquid to gas during a flash, the temperature reached at the black body will not be as high, because of the energy required for the change. Thus, reactions are usually initiated only if decomposition temperatures are below phase change temperatures. Also, as a particle gets larger, more energy is required to heat it, with a resulting lower final temperature. Matrices with better insulating properties retain the heat around the particle, tending to raise the local temperature.

Better control of black body concentration and agglomeration can be achieved by using filamentary material instead of particles, according to the authors. Fibers of tungsten and glass have been used with good results.

Many of the pyrolyzed compounds exhibit fluorescence. This is true of the areas around flashed dust spots in polyethylene, for instance, as well as with the flashed mineral oil. The structure of the fluorescent materials has not been determined, but it is felt that they are probably unsaturated linear or condensed ring compounds.

While much additional work remains to be done, this new technique promises to be of great interest to chemists studying high temperature, high speed reactions.

Ruby Maser.—Man-made rubies have been put to work at The University of Michigan in a low-temperature device—the ruby maser—which can outperform both electron tubes and transistors. From a practical point of view, the ruby maser opens up new and exciting possibilities of greatly improved civil and military communications in the microwave frequency range from 1,000 to 20,000 megacycles. It promises television reception over much longer distances, communication from space vehicles, and incorporation in advanced military surveillance devices. The technique harnesses the gyroscopic motion of electrons at temperatures near absolute zero to amplify extremely weak radio signals.

The maser is so sensitive it can detect the tiny natural radio emission from any object warmer than the ruby itself. It, therefore, can discern radiation from cool bodies a short distance away or from invisible stars which may be thousands of light years distant.

Arrangements are being made to incorporate a ruby maser in the new 85-ft. radio telescope under construction by the University at Peach Mountain, near Dexter.

Scientists at the U-M Willow Run Laboratories—Dr. Chihiro Kikuchi (Ph. D.), Dr. John J. Lambe (Ph. D.), George Makhov, and Dr. Robert W. Terhume (Ph. D.)—were the first to observe maser action in ruby. The

discovery was made in December, 1957, and is the result of investigations covering a period of more than a year.

At the high operating temperatures of ordinary vacuum tubes and even at the room temperature of transistors, atomic motion produces radio noise. In the ruby maser, at temperatures near absolute zero, this atomic noise is almost silenced. It is the reduction of this internal noise that makes possible the amplification of extremely low-level signals.

In the U-M maser system, the ruby is placed in a magnetic field and immersed in liquid helium. The magnetic field aligns the electrons of the ruby's atoms, while the frigid helium reduces the noise from the atoms' motion. Electrical energy is supplied to the ruby's electrons so that an incoming radio signal can trigger its release and produce a large signal output.

For practical applications the ruby is by far the most suitable material yet discovered, although research is being carried on throughout the country by leading university and government laboratories.

Synthetic rubies are now available in production quantities at very low cost and have extremely long life in comparison with other electronic components.

Further development work is under way to package the ruby maser in units the size of a small filing cabinet; ultimately it may be an even more compact form.

Continuous Processing of Silicone Gums and Oils.—A new catalyst and a new process have been combined experimentally at the General Electric Research Laboratory to demonstrate "continuous processing" of silicone gums and oils.

Three General Electric chemists described the discoveries in papers presented before the American Chemical Society in Chicago. They emphasized that the new continuous process was a laboratory-scale achievement and has not been fully evaluated as a possible substitute for the "batch-process" techniques now used throughout the silicone industry.

Drs. Simon W. Kantor and Alfred R. Gilbert told how they found a new "transient catalyst" that is easily neutralized after polymerization of organosiloxanes takes place. The catalyst, a strong base of the potassium hydroxide type, does an efficient job of polymerizing when heated to a moderate temperature; after heating to a slightly higher temperature it decomposes, leaving an extremely stable final product. In most silicone polymerizations, much more complicated techniques are required to remove or neutralize the catalysts.

In a second technical paper, Dr. Norman Kirk described how the transient catalyst can be used in a continuous process for making silicone gums and oils of excellent quality. Dr. Kirk's technique involves pumping the feed mixtures continuously through a heated reactor in which polymerization occurs. The polymer then flows through a second unit wherein the temperature is raised to destroy the catalyst.

Container Permits 360-Degree Inspection of Transistors.—A transparent plastic container, designed to provide greater protection and ease of handling for short-lead transistors has been introduced by the Sparks Corporation, of Philadelphia.

The new package which holds a hundred transistors can be used for shipping, storage and display of finished items or for handling of units

during the various stages of production. Because it holds each transistor individually and permits viewing of the contents from all angles, the new package simplifies many problems in inventory and inspection.

Vacuum formed from single sheets, this compact container measures $3\frac{1}{4} \times 5\frac{1}{4} \times \frac{3}{4}$ in. and is made up of two sections— a tray and a cover.

The tray is formed from .020-cellulose acetate butyrate and contains 100 pockets. These pockets act as snug-fitting individual containers for the transistors and hold them upright and securely in place so that they are easily accessible and will not move during shipping.

The cover is formed from a sheet of .015 cellulose acetate and provides a rigid top so that the boxes can be stacked without exerting pressure on the leads.

For shipping, cardboard outer protectors are supplied with each container. Both the container and the protector are available either plain or printed.

Finger-Sized Perfect Silicon Crystals.—Dislocations, the rows of out-of-line atoms that scientists now blame for making metals "much weaker than they ought to be", have been completely eliminated from crystals of silicon larger than a man's finger.

In making these "perfect" crystals in unprecedented size (previously reported perfect crystals have been whiskers thinner than a human hair) scientists at the General Electric Research Laboratory have: (1) disproved a prevalent idea that dislocations "have to be there or crystals can't grow"; (2) found important new knowledge about the causes of dislocations and how to get rid of them; (3) opened a door that may lead to making dislocation-free, and perhaps superstrong, crystals of a variety of materials in large sizes; and (4) determined that elaborate precautions will have to be taken to protect the surfaces before superstrong "perfect" crystals can be used for structural purposes; the dislocations which cause weakness apparently can be easily induced into crystals by minor surface damage.

A perfect rod of iron an inch in diameter, if it could be made, might be as strong as a huge I-beam. The catch is that even a small scratch made with a needle might induce dislocations that would propagate, when stressed, making the rod a hundred times weaker.

Nevertheless, there are many reasons why General Electric scientists are excited by the major scientific breakthrough which has produced the cigar-shaped pieces of silicon with unprecedented atomic perfection.

"We anticipate a flood of new knowledge about the electrical, mechanical, magnetic, and other properties of materials as a result of such studies," Dr. Guy Suits, General Electric vice president and director of research, said.

Discoverer of the new crystal-growing technique is Dr. William C. Dash, General Electric physicist who described the work in Cooperstown, N. Y. at an International Conference on Crystal Growth sponsored jointly by the U. S. Air Force and the General Electric Research Laboratory.

The secrets of Dr. Dash's techniques involve the use of specially tapered "seeds" made of high-perfection material. In this way he reduces the thermal shock experienced by the seed when it first touches the hot molten silicon from which the crystal is grown. Equally important is pulling the crystal in the proper crystal-

lographic direction, so that dislocations are literally "left behind" as the crystal grows.

Purity of the material is important but is not a critical problem, Dr. Dash says. His largest dislocation-free crystal, which weighs about two ounces, is similar in purity to silicon used for manufacturing transistors and other semiconductor devices—a few impurity atoms per million silicon atoms.

The strength differences between pieces of silicon with and without dislocations can be observed in a meaningful way only at temperatures of about 900 degrees C or greater. Strength-testing of the silicon crystals is difficult because of the high temperature required and the fact that anything used to grip the sample tends to induce the strength-dissipating dislocations. Nevertheless, it has been established that those without dislocations are definitely stronger at high temperatures.

Extremely strong, whisker-size crystals of iron, copper, and many other metals have been made at the General Electric Research Laboratory and at several other laboratories throughout the world. In the case of iron, strengths up to nearly two million pounds per square inch have been observed, five times the tensile strength of the strongest piano wire. As a result of Dr. Dash's studies, there is the prospect of making large size perfect crystals of many substances with high strength—provided that the surfaces are protected.

The absence of dislocations in Dr. Dash's large silicon crystals has been ascertained in a variety of ways: (1) by surface observations of a chemically etched surface; (2) by "decoration," a method in which copper is precipitated along the dislocations inside the silicon crystal; the dislocations, or lack of them, can then be observed by "looking through" the silicon, which is transparent to infrared light; and (3) by X-ray inspection.

While work is being directed toward growing dislocation-free crystals of a variety of metals, the only other material in which results have been observed similar to those in silicon is germanium, also a semiconductor from which transistors and other devices are made.

Dr. Arthur G. Tweet, another Research Laboratory physicist, has used the same technique to grow crystals of germanium that are free of dislocations.

The role of dislocations in determining the strength of crystalline solids has been part of new theories evolving during recent decades. It has been shown that plastic deformation of metals—bending them so they stay bent—is accomplished when dislocations permit planes of atoms to slide over each other. In the absence of dislocations, the theory suggests that metals can be bent further elastically—they return to their original shape after greater forces have been applied. The extent of such elastic deformation is a principal scientific measure of strength.

Crude Oil Analysis Using Nuclear Reactor.—Unwanted elements present in crude oils are being forced to make their own "fingerprints" at The University of Michigan. The investigation, utilizing facilities of the Michigan Memorial-Phoenix Project, is aimed at permitting speedier and more accurate detection of elements such as vanadium to prevent their "poisoning" certain steps involved in petroleum refining. The work is under the direction of W. Wayne Meinke, associate professor of chemistry.

Playing key roles in the research are the Project's Ford Nuclear Reactor, most powerful at any educational institution, and a unique 100 channel analyzer, a specially prepared instrument which draws a graph picture of the element on "instructions" from the element.

The process also has reduced from a day to sometimes as little as a few minutes the time necessary to make such analyses. The amount of the elements present in the crude oils is so small that the usual chemical steps involved in measuring them not only are too time consuming but also tend to depreciate in value because the supposedly pure test chemicals themselves contain enough of the unwanted elements to cloud the results.

At the University, a sample of the crude oil is sent through a pneumatic tube to the core of the reactor where in minutes it becomes radioactive. Then it is returned to the chemist via the pneumatic tube in just three seconds. Immediately placed in the analyzer, the sample, through its radioactivity, influences the instrument to draw a graph which will "fingerprint" the element under investigation and will indicate the amount of the element in the crude oil. By comparing the resulting "fingerprint" with graphs taken previously when the element was known to be present, chemists can tell if the element is in the sample and, if so, how much.

The analyzer differs radically from any others of its type in that it is equipped with dual "memory" to store up the radioactive impulses it gets from a sample during the short period of time desired radioactivity is present. This guarantees that the graph-making portion of the instrument will be able to "fingerprint" the element at its more leisurely pace, one memory instructing the graph-maker while the other memory stores information. Operating on a pre-set cycle, the memories then change "places," and the first, which has passed along all its information, stores up new, while the second begins instructing.

Plant Classification Using Data-Processing System.—What may be one of the most significant developments in plant classification techniques since Linnaeus founded his binomial cataloguing system was revealed at the meeting of the American Institute of Biological Sciences, held in Bloomington, Ind.

The new technique uses a large-scale IBM 704 data processing system to classify plants automatically according to their individual characteristics. It was developed by Dr. D. J. Rogers of the New York Botanical Gardens and Dr. T. T. Tanimoto of the Mathematics and Applications Department of International Business Machines Corporation.

The new technique begins with an exacting description of the plant. Up to 100 characteristics, including size, color, root structure, leaf structure, and stem structure, are recorded in IBM punched cards. Decks of these cards are then fed into the IBM 704 computer. The 704 digests this vast assemblage of data and automatically classifies the plants into natural groups or clusters, with each cluster being determined by the over-all physical similarities of the plants that comprise it.

Because the computer analyzes such a great number of characteristics, it can detect subtle similarities linking plants to certain groups which otherwise might not be apparent even to a skilled botanist.

In a paper delivered at the meeting, Dr. Rogers and Dr. Tanimoto described the application of the new computer technique to the classification of 100 varieties of *Manihot esculenta*, the tropical food plant whose edible rootstocks form the basic diet of the tribal Indians of South America. Punched cards containing data on fifty physical characteristics of each variety of *Manihot* are fed into the IBM 704. The computer analyzes these characteristics and classifies the varieties into natural groups. The classification arrived at by the computer could also be used to ascertain the various physical characteristics of the *Manihot* that are closely related to high food yield, which in turn could prove useful in the development of higher yield varieties.

The Drs. Rogers and Tanimoto emphasized that their new plant classification technique could also be applied to other areas where the classification of objects is desirable in terms of the over-all similarities of characteristics. For example, in the medical field, diseases might be automatically classified on the basis of similarity of various measurable symptoms.

Although the new classification technique was developed specifically for use with IBM's 704 computer, it was pointed out that the same principle can be used with any large-scale computing system.

Britain Opens First Automatic Telex Exchange.—Britain's first automatic telex exchange went into operation in Leeds, England, on September 1. By the end of September about 1,000 subscribers in this area and in another new automatic exchange area—Shoreditch, London—were able to reach other "automatic" subscribers by simply dialing a number. Conversion to automation of all of the country's 50 telex centers will be completed early in 1961. Moreover, the cost of 90 per cent of the calls within Britain will go down as a result of tariff concessions coupled with a new method of charging that goes into effect with the changeover to automation.

Telex is a system of telegraphic communication operated between the premises of business organizations in the United Kingdom and throughout the world. It combines the speed of the telephone with the authority of the printed word. A printed copy of the communication is produced on teleprinters at both the sending and receiving subscribers' installations. The service is available day and night, enabling messages to be transmitted to a subscriber even though his teleprinter is unattended. The message will then be available when the office opens.

Under the manual system, the charge within Britain for the minimum three-minute call has varied from 6d. to 2s. (12¢ to 28¢), according to the distance involved. Now, subscribers dialing their own calls will buy time in twopennyworths (2⅓ cents). The amount of time that can be bought for twopence (2⅓ cents) will depend on the distance of the call. For instance, for twopence a subscriber on an automatic telex exchange will be able to buy one minute of time on local calls and on calls to telex centers up to 35 miles away. For that same money, he will have 30 seconds on calls between 35–50 miles; 20 seconds on those from 50–75 miles and 15 seconds on those over 75 miles. An experienced operator can send up to 60 words a minute, so that in effect a 60-word message can now be sent as far as 35 miles for 2d.

Charges for calls obtained automatically will be recorded by meter, and the caller will pay (according to

the distance called) only for the time actually used. Since many inland calls are brief—shorter than the three minute minimum formerly charged for—a high percentage of calls will cost less. Eventually, a subscriber will be able to rent a meter for installation at his premises.

A new type of public call office enabling private callers to teleprint messages personally to telex system subscribers throughout the world was started experimentally last year in London and Manchester. Callers operate the machines personally by calling the telex switchboard to get the required connection and then teleprinting their messages. Afterwards, they pay their fee to an attendant. Journalists, travelers and businessmen find this service particularly useful. Often it is cheaper than dictating messages slowly over the 'phone. They can be sure their actual words are received in print, and the caller himself has a copy of his message.

Britain's telex system—run by the Post Office, like the telephone system—grows in value in proportion to the number of its subscribers. There were 20 subscribers during 1947, the year when the service was introduced into Britain; now there are 4500.

Last year, more than 3,000,000 telex calls representing over 13½ million paid minutes of traffic were exchanged between the United Kingdom and countries overseas, compared with 25,800 calls representing 158,000 paid minutes in 1947.

British subscribers vary from business firms (a high percentage are engaged in international trade, manufacture of engineering products, distribution and transport, including shipping) to banking and finance companies and large hotels.

World-wide telex subscribers total more than 80,000, of whom 42,000 are in the U. S. A. The telex service is available in North and South America, South Africa, the Federation of Rhodesia and Nyasaland and in many other countries as far apart as Norway, Hawaii, the Philippines, Yugoslavia and Morocco. Nationally and internationally, telex is steadily achieving the object of developing a self-contained communications system for the whole worldwide business community.

New Drug for Skin Diseases.—A potent new hormone drug that may be applied directly to the skin has shown dramatically effective results in the treatment of various skin diseases. The chemical name for the new steroid substance, which was developed by The Squibb Institute for Medical Research, is triamcinolone acetonide. Its trade name is Kenalog.

In clinical studies of the new drug, involving more than 1000 patients with various skin disorders, unusually effective results were produced in more than 95 per cent of the study group. For example, in one clinical trial in which 18 patients were treated with Kenalog, the reporting physicain stated that the drug "proved to be excellent or even a superb agent in these patients."

Another clinician, reporting results on the use of Kenalog with 65 patients with a variety of skin disturbances, said 62 of these people obtained excellent to good results. The patients, as well as their personal physicians, in some instances described the results as "dramatic" or even "miraculous."

Among the skin disorders against which Kenalog was found to be particularly effective were atopic dermatitis, contact dermatitis, eczema, neurodermatitis, seborrheic dermatitis and a number of other inflammatory conditions.

In addition to the visible results obtained with Kenalog, the drug also gives rapid and complete relief of itching and burning and other skin discomfort. Unlike some other hormonal preparations, Kenalog is remarkably free of side effects. Substantially less than one per cent of patients treated with the drug complained of any burning or stinging.

The new product, a derivative of the steroid triamcinolone, was made possible by a major scientific breakthrough in the chemistry of the naturally-occurring hormone, hydrocortisone. Hydrocortisone is a steroid. Steroids are a group of chemicals that have been proven useful in treating arthritis and allied conditions. Because the hydrocortisone molecule is so specifically engineered by nature for the job it does, it seemed hopeless to try to improve upon it. But it was improved by an ingenious rearrangement of the molecular make-up of the hormone. This was accomplished by scientists at The Squibb Institute for Medical Research. The result was a made-to-order molecule—fluorohydrocortisone—ten times as potent as the natural substance. Now, as a further outgrowth of this advance, Squibb is making available triamcinolone acetonide, which is 40 times as potent as hydrocortisone and substantially more potent than fluorohydrocortisone.

Available only upon doctors' prescription, Kenalog comes in three forms: cream, lotion and ointment.

Miniaturized Aircraft Mass Flowmeter.—A new miniaturized aircraft mass flowmeter system that will provide a continuous indication of the mass rate of fuel flow up to 5000 lb. per hour has been developed by General Electric's Instrument Department, West Lynn, Mass. Designated type TJ-61, it is designed for use on small jet and turbo-prop engines.

A self-contained frequency compensator in the compact self-synchronous system permits operation from 400 cycle, 115-volt ship's power which does not have close frequency control. No external power converter is needed, even for applications in which a controlled frequency power source is unavailable.

Instrument Department engineers said that the system is now in design and tooling and will be in production by the second quarter of 1959.

As a safety measure, all electrical components and all wires are sealed off from the fuel, and the flowmeter is electrically grounded to the frame of the aircraft.

The flowmeter transmitter weighs 2.5 lb. and is only 4½ in. long. It will withstand severe mechanical abuse and will operate within a wide temperature range—minus 55 to plus 121°C. Proved characteristics of the measuring elements minimize error due to such ambient conditions as temperature and vibration and to such fuel properties as density and viscosity.

The transmitter can be furnished with General Electric's new DJ-96 fuel flow rate indicator. True mass rate of flow is indicated on its integrally lighted pointer and dial scale. The transmitter can also be furnished with a new fuel-consumed indicator, Type DJ-77. All transmitter bearings are designed for durability and low friction.

This new miniaturized flowmeter system represents one of the latest advances in General Electric's development work—begun nine years ago—on methods for direct measurement of fluid mass.

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